Few remarks on wave eqn and Fourier transform

$\partial_t^2 \varphi = v^2 \partial_x^2 \varphi$ general solution is $\varphi = f(x-vt) + g(x+vt)$

We may decompose $f(x)$ and $g(x)$ in Fourier space

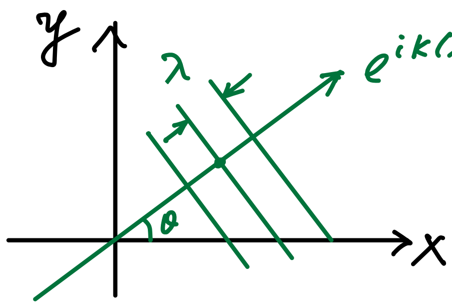
$$f(x) = \int f(k) e^{ikx} dk \Rightarrow f(x-vt) = \int f(k) e^{ik(x-vt)} dk$$

$$\boxed{\text{Initial condition } \varphi(x,0) = \int \underbrace{f(k)}_{\text{forward}} e^{ikx} + \underbrace{g(k)}_{\text{backward}} e^{ikx} dk \quad \text{moving}}$$

$$\boxed{\text{Late time solution: } \varphi(x,t) = \int \underbrace{f(k)}_{\text{amps}} \underbrace{e^{ik(x-vt)}}_{\text{basis}} + \underbrace{g(k)}_{\text{amp}} \underbrace{e^{ik(x+vt)}}_{\text{basis}} dk}$$

also called plane wave

What if the wave is propagating in other directions?



$$e^{ik(x'-vt)} = e^{ik(x\cos\theta + y\sin\theta - vt)}$$
$$= e^{i(k\cos\theta\, x + k\sin\theta\, y - vt)}$$
$$\equiv e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

$\boxed{\text{Wave number } \vec{k} = (k_x, k_y) \text{ is a vector in the propagation direction.}}$

$\Rightarrow e^{i(\vec{k}\cdot\vec{r} - \omega t)}$ describes wave moving in $\vec{k}$ with wave # $|\vec{k}| = 2\pi/\lambda$

$\Rightarrow \partial_t^2 \varphi = v^2(\partial_x^2 + \partial_y^2)\varphi$

$\boxed{\partial_t^2 \varphi = v^2 \nabla^2 \varphi}$ in 2D-dimensions.

Double check: $\partial_t^2 \varphi = \omega^2 \varphi$

$\partial_x^2 \varphi + \partial_y^2 \varphi = (\cos^2\theta + \sin^2\theta) k^2 \varphi$

$\Rightarrow \omega^2 = v^2 k^2$

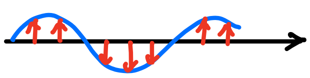
Wave phenomena : longitudal and transverse waves.

Longitudinal

propagation Displacement along the propagation

Examples : sound, spring. seismic P-wave

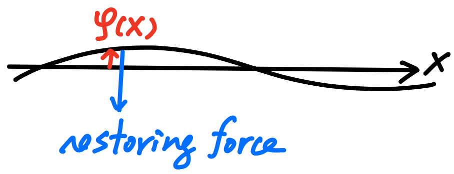
Transverse

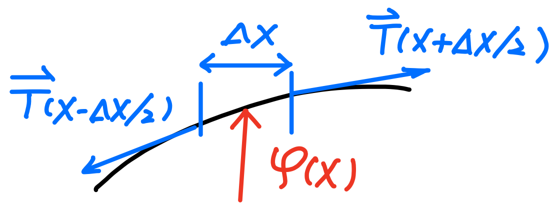


Displacement transverse to the propagation

Examples : Strings. EM waves. seismic S-wave. water waves

Strings





Tension $\vec{T} = (T_x, T_y)$ is along the string $T_y/T_x = \varphi'(x)$

$$\Rightarrow \vec{T}(x-\Delta x/2) = -(T_1, T_1\varphi'(x-\Delta x/2))$$
$$\vec{T}(x+\Delta x/2) = (T_2, T_2\varphi'(x+\Delta x/2))$$

no transvese motion : $\sum \vec{T}_x = T_2 - T_1 = 0$

yes longitudinal motion: $\sum \vec{T}_y = T\varphi'(x+\Delta x/2) - T\varphi'(x-\Delta x/2) = \rho\Delta x\,\partial_t^2\varphi$

$$\Rightarrow \rho\Delta x\,\partial_t^2\varphi = T(\varphi'(x+\Delta x/2) - \varphi'(x-\Delta x/2))$$
$$= T\Delta x\,\partial_x^2\varphi$$

$\Rightarrow \boxed{\rho\partial_t^2\varphi = T\partial_x^2\varphi}$ Compared with $\partial_t^2\varphi = v^2\partial_x^2\varphi$

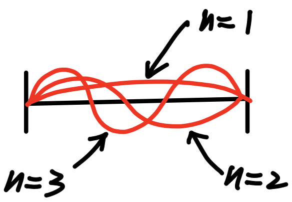
$\Rightarrow$ String velocity $\boxed{v = \frac{\omega}{k} = \sqrt{T/\rho}}$

If length is fixed k=const. higher tension higher pitch $\omega \propto \sqrt{T}$

higher density lower pitch $\omega \propto \sqrt{1/\rho}$

Boundary conditions

1. Fixed $\varphi(x=0)=\varphi(x=L)=0$



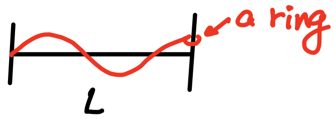
$$\varphi(x)=\sum_{n=1}^{\infty} a_n \sin n\frac{\pi x}{L}$$

How about time dependence? Each wave goes like $e^{\pm ikx}e^{\pm i\omega t}$ with $v=\omega/k$ given by $\sqrt{T/\rho}$

$$\varphi(x,t)=\sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}\left(b_n e^{ikvt}+c_n e^{-ikvt}\right)$$

$$=\sum_{n=1} \sin\frac{n\pi x}{L}\left(b_n' \cos\frac{n\pi}{L}vt + c_n' \sin\frac{n\pi}{L}vt\right)$$

2. Open boundary condition



$L=\frac{1}{4}\lambda, \frac{3}{4}\lambda, \frac{5}{4}\lambda \cdots$

$=\frac{2n+1}{4}\lambda \quad n=0,1,2\cdots$

another way to write the solution

$$\varphi=\sum_{n=0} a_n \sin k_n x \cos(\omega_n t+\phi_n)$$

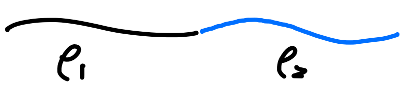
$$k_n=\frac{2\pi}{\lambda}=2\pi\frac{2n+1}{4L}$$

3. Free boundary condition

$$\varphi=\sum_n a_n e^{ik_n(x-vt)}+b_n e^{ik_n(x+vt)} \rightarrow \int \varphi_+(k)e^{ik(x-vt)}+\varphi_-(k)e^{ik(x+vt)}dk$$

forward waves   backward waves.

4. Connecting two strings of diff. density ρ_1, ρ_2



Waves going to diff medium
water, light, seismic and others.

What do we do @ the boundary?