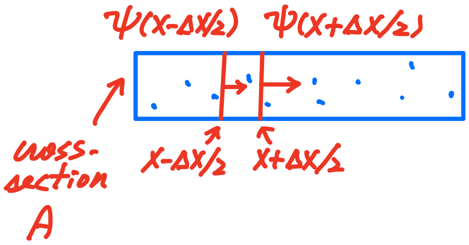
Longitudinal waves: Sound wave in gas, liquid, and solid.

step 1: What is the equilibrium state?

step 2: What's the restoring force?

step 3: Associate the force with the motion of neighbors.

Consider a uniform medium with mass density n



$$ma = n\Delta V \partial_t^2 \varphi = nA\Delta x\, \partial_t^2 \varphi$$

$$F = AP(x-\Delta x/2) - AP(x+\Delta x/2)$$

$$= -A\Delta x P'(x)$$

In gas and liquid $P>0$

In solid $P<0$

What is the origin of restoring force?

Let's introduce compressibility $\beta \equiv -\frac{1}{V}\partial V/\partial P \quad \left(= \frac{1}{\text{stiffness}}\right)$

$$\Delta P = -\frac{1}{\beta}\frac{\Delta V}{V} = -\frac{1}{\beta}\frac{A\left[\varphi(x+\Delta x/2) - \varphi(x-\Delta x/2)\right]}{A\Delta x}$$

pressure in equilibrium

$$\Rightarrow \boxed{P(x) - P_0 = -\frac{1}{\beta}\varphi'(x)}$$

$$n = \frac{n_0 V}{V+\Delta V} = n_0\left(1+\frac{\Delta V}{V}\right)^{-1} \approx n_0\left[1-\varphi'(x)\right]$$

Thus we have $nA\Delta x\, \partial_t^2\varphi = -A\Delta x\left(-\frac{1}{\beta}\partial_x^2\varphi\right)$

$$\Rightarrow n\partial_t^2\varphi = \frac{1}{\beta}\partial_x^2\varphi \quad \text{or} \quad \partial_t^2\varphi = v^2\partial_x^2\varphi$$

$$\Rightarrow \boxed{v = \sqrt{1/n\beta}}$$

For an ideal gas:

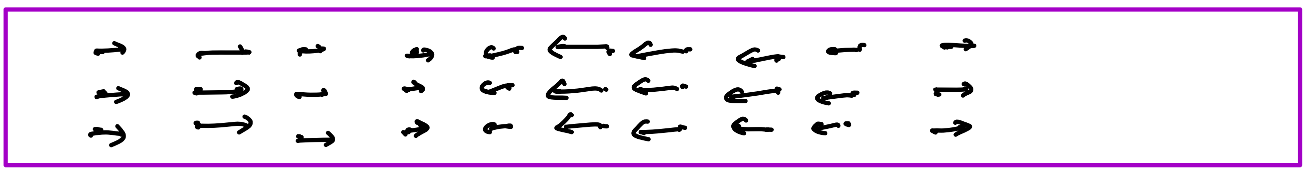
$PV = NkT$. For T = const. $\beta = -\frac{1}{V}\partial V/\partial P = \frac{1}{V}\frac{NkT}{P^2} = \frac{1}{nkT}$

$$\Rightarrow v = \sqrt{kT} = \sqrt{\partial P/\partial n}$$

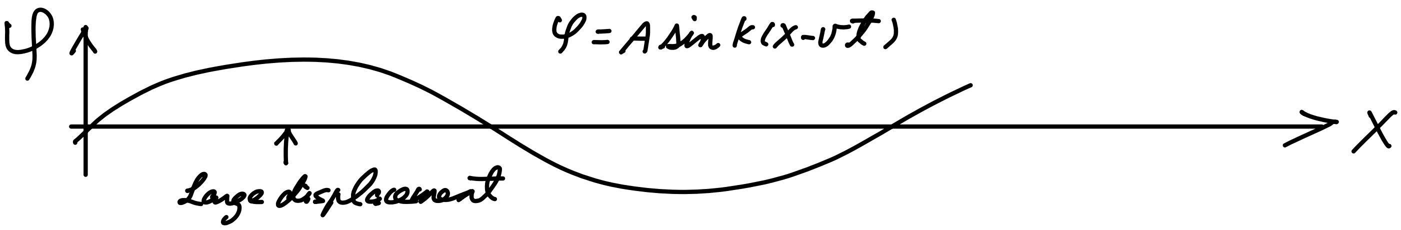
Faster sound for lower density and lower compressibility

Let's compare	air $^{340\,m/s}$	water $^{1500\,m/s}$	ice $^{4000\,m/s}$
n	small	high	higher
β	high	low	lower

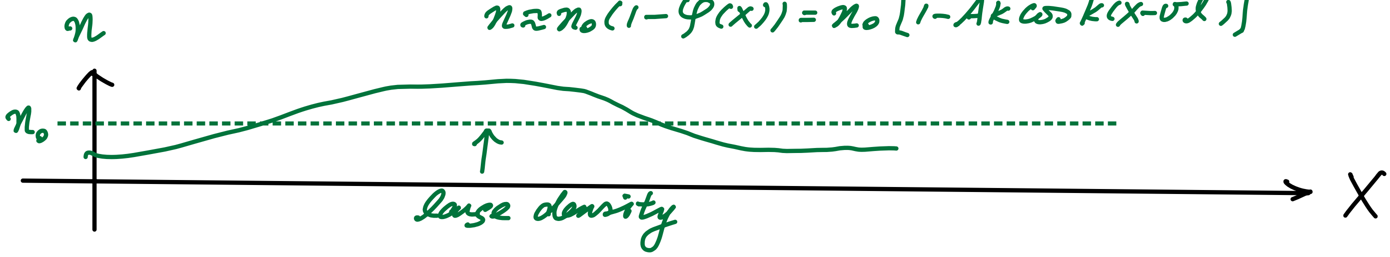
Microscopic picture



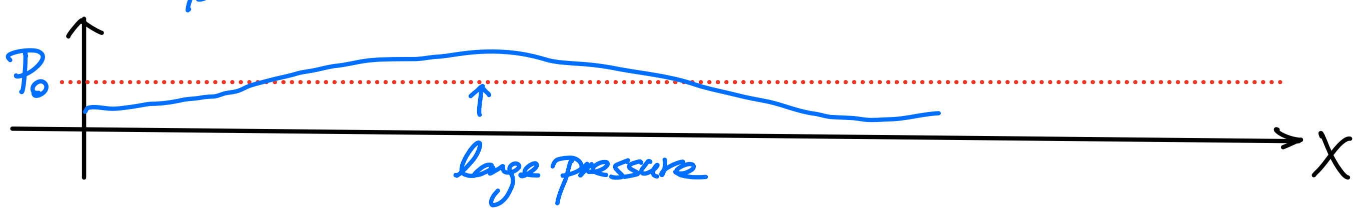
$$\varphi = A \sin k(x - vt)$$



$$n \approx n_0 (1 - \varphi'(x)) = n_0 [1 - Ak \cos k(x - vt)]$$



$$P = P_0 - \frac{1}{\beta} k \cos k(x - vt)$$



Electromagnetic waves

Maxwell Eqns in medium

$\nabla\cdot E = \rho/\epsilon = 0$ Gauss' law

$\nabla\cdot B = 0$ no mag. monopole

$\nabla\times E = -\partial_t B$ Faraday's law

$\nabla\times B = \mu\epsilon\,\partial_t E + \mu_0 J = \mu\epsilon\,\partial_t E$ Amp-Max. law

$$\left.\begin{aligned}\nabla\times(\nabla\times E) &= \nabla(\nabla\cdot E) - (\nabla\cdot\nabla)E = -\nabla^2 E\\ &= -\partial_t \nabla\times B = -\mu\epsilon\,\partial_t^2 E\end{aligned}\right\}\ \partial_t^2 E = \frac{1}{\mu\epsilon}\nabla^2 E \qquad v = \frac{1}{\sqrt{\mu\epsilon}}$$

Similarly $\partial_t^2 B = \frac{1}{\mu\epsilon}\nabla^2 B$

In vacuum $\partial_t^2 \begin{Bmatrix} E \\ B\end{Bmatrix} = \frac{1}{\mu_0\epsilon_0}\begin{Bmatrix} E \\ B\end{Bmatrix}$ $v = \frac{1}{\sqrt{\mu_0\epsilon_0}} = c$ $\Rightarrow$ index of refraction $n = \frac{c}{v} = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}}$

We may think the solution is $\vec{E}(\vec{x},t) = \Sigma_k \vec{A}(\vec{k}\cdot\vec{x} - \omega_k t)$ $\omega_k = v|\vec{k}|$

But the other Maxwell eqns provide further restriction

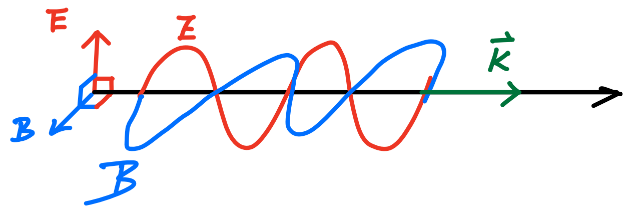
$\vec{E}(x,t) = \vec{E}\,e^{ik(x-ct)} = E_j\,\hat{e}_j\,e^{ik_\ell x_\ell}\,e^{-i\omega t}$

Gauss' law

$\nabla\cdot E = \partial_j E_j e^{ik_\ell x_\ell} e^{-i\omega t} = \vec{k}\cdot\vec{E} = 0.$

$\nabla\cdot B = \partial_j B_j e^{ik_\ell x_\ell} e^{-i\omega t} = \vec{k}\cdot\vec{B} = 0$

Faraday's law $\nabla\times E = -\partial_t B \Rightarrow i\vec{k}\times\vec{E} = -(-i\omega)\vec{B} \Rightarrow \hat{k}\times\vec{E} = v\vec{B}$



Forward $E^+ = (E, 0, 0)\,e^{ik(z-ct)}$

$B^+ = (0, E/c, 0)\,e^{ik(z-ct)}$

backward $E^- = (E, 0, 0)\,e^{ik(-z-ct)}$

$B^- = (0, -E/c, 0)\,e^{ik(-z-ct)}$

Standing waves: $E = E^+ + E^- = (E, 0, 0)[\cos k(z-ct) + \cos k(-z-ct)]$

$= (2E, 0, 0)\cos kz \cos\omega t$

$B = B^+ + B^- = (0, 2E/c, 0)[\cos k(z-ct) - \cos k(-z-ct)]$

$= (0, 2E/c, 0)\sin kz \sin\omega t$