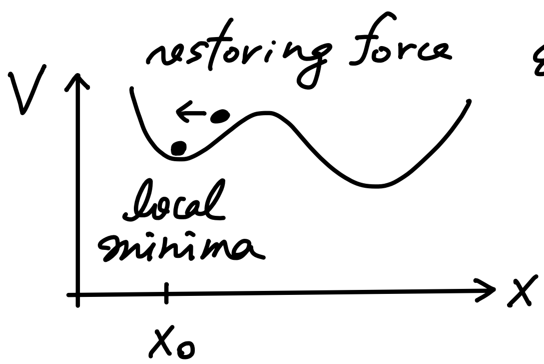
Lecture 1~3 Harmonic oscillator Cheng Chin

This class concerns motion of particles near equilibrium.

Consider one particle with mass m in a potential $V(x)$



Equilbrium @ $X=X_0$

$$\Rightarrow F(X=X_0)=-\frac{d}{dx}V(X_0)=0$$

$$\Rightarrow V'(X_0)=0$$

We may Taylor expand the potential @ $X=X_0$

$$V(x)=V(X_0)+\underbrace{V'(X_0)}_{=0}(X-X_0)+\frac{1}{2}V''(X_0)(X-X_0)^2+\frac{1}{6}V'''(X_0)(X-X_0)^3$$

If particle is near the minima $X\simeq X_0$ we ignore $(X-X_0)^3$

$V(X)\simeq V(X_0)+\frac{1}{2}V''(X_0)(X-X_0)^2$ is a good approximation

Math notation:

velocity $v=\frac{dX(t)}{dt}\equiv X'(t)\equiv\dot{X}(t)$

acceleration $a=\frac{dv(t)}{dt}$

$$=\frac{d}{dt}\frac{dX(t)}{dt}$$

$$=\frac{d^2X(t)}{dt^2}$$

$$\equiv X''(t)\equiv\ddot{X}(t)$$

$$\equiv v'(t)\equiv\dot{v}(t)$$

We may set the origin @ $X = X_0 \Rightarrow V(x) = V(0) + \frac{1}{2}V''(0)X^2$

Eq. of motion $F = ma$

$$\Rightarrow mx''(t) = F = -\frac{d}{dx}V(x) \Rightarrow mx''(t) = -\frac{d}{dx}\left[V(0) + \frac{1}{2}V''(0)X^2\right]$$

$$\Rightarrow mx'' = -V''(0)X$$

$$\Rightarrow mx'' + kX = 0.$$

This is equivalent to Hooke's law $F = -kx$ and $k = V''(0) \Rightarrow$ positive curvature ensures restoring force and stable eq.

If we add a damping force $F = -\beta v$, $\beta > 0$ is damping coefficient.

we have $mx'' = \underbrace{-kx}_{\text{restoring force}} \underbrace{-\beta x'}_{\text{damping force}} \Rightarrow mx'' + \beta x' + kx = 0$

If we add an external driving force $f(t) \Rightarrow mx'' = -kx - \beta x' + f(t)$

we have $mx'' + \beta x' + kx = f(t)$. This is the eqn we will be solving.

See videos for solving ODE ordinary differential eqn.

$$Ay''(x) + By'(x) + Cy(x) = 0 \quad \text{homogeneous ODE}$$
$$= f(x) \quad \text{inhomogeneous ODE}$$

Important remark on linear homogeneous diff. eqn

If $y_1(x), y_2(x) \cdots$ are solutions.

Then $Ay_1(x) + By_2(x) \cdots$ is also a solution

Simple harmonic oscillator (no damping, no ex. force)

$mx'' + kx = 0$

Ansatz for ODE: $x = e^{\alpha t}$ thus $x' = \alpha e^{\alpha t}$, $x'' = \alpha^2 e^{\alpha t}$

$\Rightarrow m\alpha^2 \cancel{e^{\alpha t}} + k\cancel{e^{\alpha t}} = 0 \Rightarrow \alpha^2 = -\frac{k}{m} \Rightarrow \alpha = \pm i\sqrt{\frac{k}{m}} \equiv \pm i\omega$

$\Rightarrow x = e^{i\omega t}$ and $e^{-i\omega t}$ are both solutions

General solution is $x = Ae^{i\omega t} + Be^{-i\omega t}$

2nd Ansatz: $x = \cos\alpha t$ or $\sin\alpha t$

$\Rightarrow x' = -\alpha \sin\alpha t$ $\quad \cos\alpha t$

$\Rightarrow x'' = -\alpha^2 \cos\alpha t$ $\quad \alpha^2 \sin\alpha t$

$\Rightarrow -m\alpha^2 \begin{Bmatrix}\cos\alpha t\\ \sin\alpha t\end{Bmatrix} + k\begin{Bmatrix}\cos\alpha t\\ \sin\alpha t\end{Bmatrix} = 0$

$\Rightarrow \alpha^2 = \frac{k}{m} \Rightarrow \alpha = \sqrt{\frac{k}{m}} = \omega$

General solution $x = C\cos\omega t + D\sin\omega t$

3rd ansatz $x(t) = A\cos(\alpha t + \phi)$,

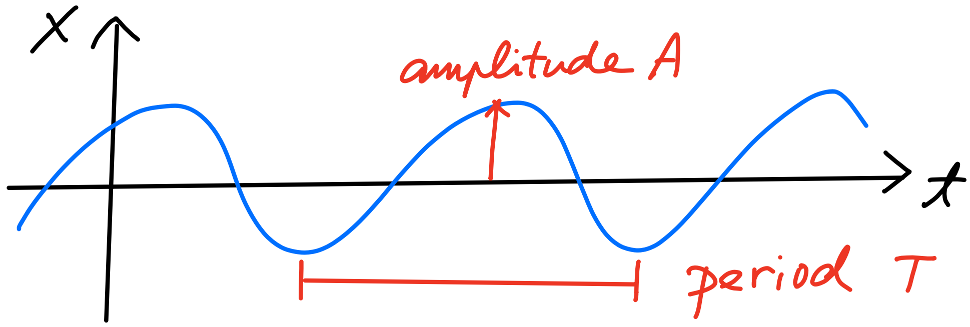
which is equivalent to $A\cos\alpha t + B\sin\alpha t$

We can see they are equivalent since $e^{\pm i\omega t} = \cos\omega t \pm i\sin\omega t$

Show that $C = A + B$, $D = iA - iB$

We can determine the integration constants $A, B / C, D$ from the initial position $x(0)$ and velocity $v(0)$.

General solution (no damping, no ext. force)



freq $f \equiv \frac{1}{T}$ angular freq $\omega = \frac{2\pi}{T} = 2\pi f$

We can also rewrite the eq. of motion and solution as

$$\boxed{\begin{aligned} &x''(t) + \omega^2 x(t) = 0 \\ &x(t) = A e^{i\omega t} + B e^{-i\omega t} \\ &\quad \text{or } A' \cos \omega t + B' \sin \omega t \\ &\quad \text{or } A'' \cos(\omega t + \phi) \end{aligned}}$$

Damped harmonic oscillator

$$x'' + \gamma x' + \omega_0^2 x = 0 \qquad \omega_0^2 = \frac{k}{m}, \ \gamma = \frac{\beta}{m}$$

Ansatz $x = e^{\alpha t}$ $\quad x' = \alpha e^{\alpha t}$ $\quad x'' = \alpha^2 e^{\alpha t}$

$$\Rightarrow \alpha^2 \cancel{e^{\alpha t}} + \alpha\gamma \cancel{e^{\alpha t}} + \omega_0^2 \cancel{e^{\alpha t}} = 0 \qquad \alpha^2 + \gamma\alpha + \omega_0^2 = 0$$

$$\Rightarrow \alpha_\pm = \frac{1}{2}\left(-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}\right) \Rightarrow x = Ae^{\alpha_+ t} + Be^{\alpha_- t}$$

Case 1: 2 real roots $\gamma > 2\omega_0$ $\Leftarrow$ Strong damping.

$\alpha_\pm \in \mathbb{R}$ and $\alpha_\pm < 0$. define $\gamma_\pm = -\alpha_\pm > 0$.

$\Rightarrow x(t) = Ae^{-\gamma_1 t} + Be^{-\gamma_2 t}$ damping, no oscillation

This is an $\boxed{\text{overdamped oscillator}}$

Case 2: $\gamma < 2\omega_0$ weak damping $\Rightarrow$ 2 complex roots.

$$\alpha_\pm \equiv -\frac{1}{2}\gamma \pm i\sqrt{\omega_0^2 - \gamma^2/4} \equiv \tilde{\omega}$$

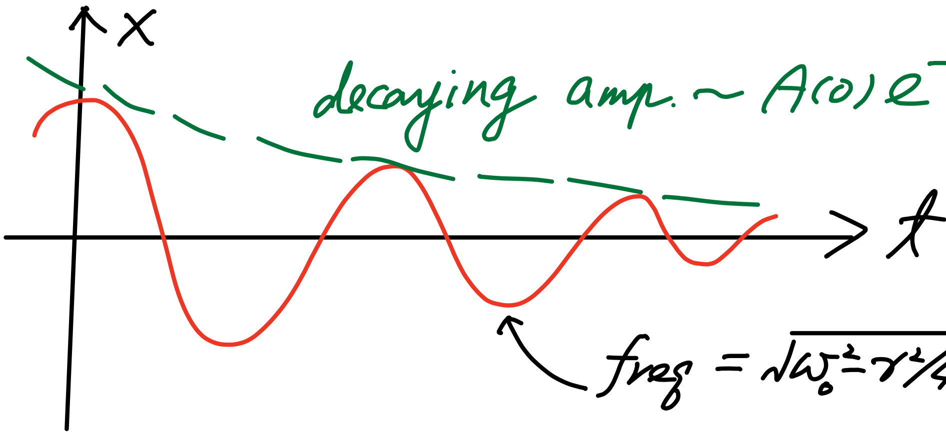
$$\Rightarrow x = Ae^{-\gamma t/2} e^{i\tilde{\omega} t} + Be^{-\gamma t/2} e^{-i\tilde{\omega} t}$$

$$= e^{-\gamma t/2}\left(Ae^{i\tilde{\omega} t} + Be^{-i\tilde{\omega} t}\right)$$

or $e^{-\gamma t/2}(C\cos\tilde{\omega} t + D\sin\tilde{\omega} t)$

or $e^{-\gamma t/2} E\cos(\tilde{\omega} t + \phi)$

This is an $\boxed{\text{underdamped oscillator}}$



decaying amp. $\sim A(0)e^{-\gamma t/2}$

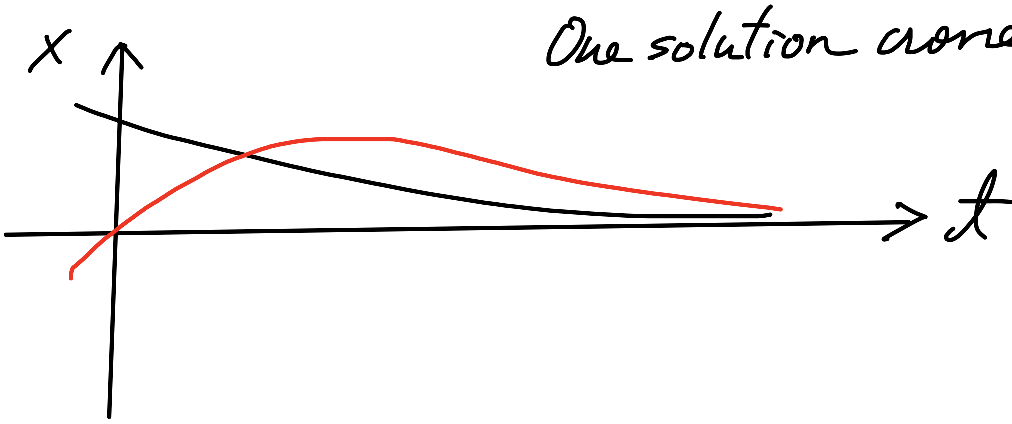
freq $= \sqrt{\omega_0^2 - \gamma^2/4} \xrightarrow[\gamma \to 0]{} \omega_0\left(1 - \frac{1}{8}\frac{\gamma^2}{\omega^2}\right)$

Case 3: $\gamma = 2\omega_0$ critical damping

$\alpha_\pm = -\gamma/2$ double root!! We are missing another sol.

$$x = Ae^{-\gamma t/2} + Bte^{-\gamma t/2}$$

One solution crosses zero, one doesn't



Driven oscillator ($\cos\omega t$ driving)

$$x'' + \gamma x' + \omega_0^2 x = f(t) \equiv f\cos\omega t$$

ω: ext driving freq
ω_0: oscillator freq

First consider $f(t) = fe^{i\omega t}$

Use ansatz $x = Ae^{i\omega t}$, $x' = iA\omega e^{i\omega t}$, $x'' = -A\omega^2 e^{i\omega t}$

$$\Rightarrow -A\omega^2 \cancel{e^{i\omega t}} + iA\omega\gamma \cancel{e^{i\omega t}} + \omega_0^2 A \cancel{e^{i\omega t}} = f \cancel{e^{i\omega t}}$$

$$\Rightarrow A(-\omega^2 + i\omega\gamma + \omega_0^2) = f$$

$$\Rightarrow A = \frac{f}{\omega_0^2 - \omega^2 + i\gamma\omega} \Rightarrow x = \frac{fe^{i\omega t}}{\omega_0^2 - \omega^2 + i\gamma\omega}$$

Thus $x'' + \gamma x' + \omega_0^2 x = fe^{i\omega t}$, we take the real part

$(\mathrm{Re}\, x)'' + \gamma(\mathrm{Re}\, x)' + \omega_0^2(\mathrm{Re}\, x) = f\cos\omega t$. So the solution of the original eqn is just $\mathrm{Re}\, x = \mathrm{Re}\, \frac{e^{i\omega t}}{\omega_0^2 - \omega^2 + i\gamma\omega} f$

We can write $\omega_0^2 - \omega^2 + i\gamma\omega = re^{i\phi}$

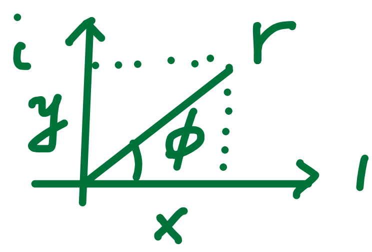
$$r = \sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}$$

$$\tan\phi = \frac{\gamma\omega}{\omega_0^2 - \omega^2}$$

$$\Rightarrow \mathrm{Re}\, x = f\,\mathrm{Re}\, \frac{1}{r} e^{-i\phi} e^{i\omega t}$$

$$= \frac{f}{\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}} \cos(\omega t - \phi) \equiv x_p$$

$z = x + iy = re^{i\phi}$



$r^2 = x^2 + y^2$, $\tan\phi = y/x$

This is the "particular solution".

General solution is $x = \underbrace{x_H}_{\text{homogeneous solution}} + \underbrace{x_p}_{\text{special solution}}$

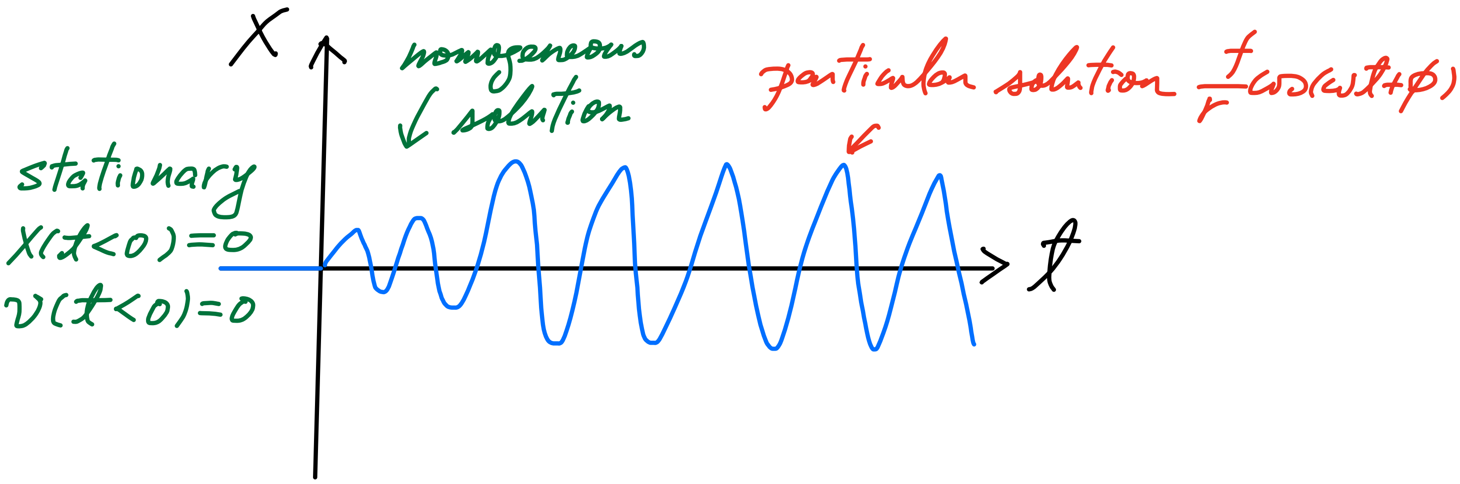
$\hat{D} x_H = 0$
$\hat{D} x_p = f(t)$
$\Rightarrow \hat{D}(x_p + x_H) = f(t)$

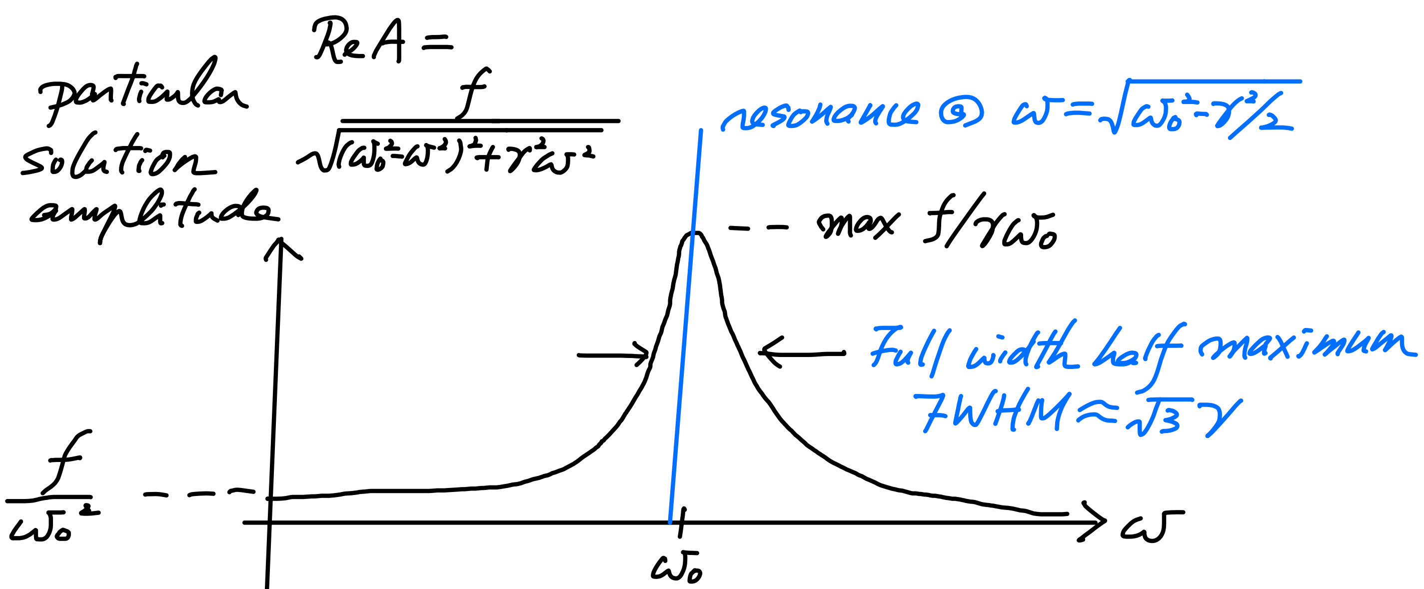
Summary $x'' + \gamma x' + \omega_0^2 x = f\cos\omega t$

Solution $x = x_H + x_p = Be^{\alpha_+ t} + Ce^{\alpha_- t} + \frac{f}{r}\cos(\omega t + \phi)$

When $\gamma > 0$, x_H damps out, solution $\to$ particular solution x_p.

Example: Starting driving at $t=0$ with force $f\cos\omega t$





Quality factor $Q = \omega_0/\gamma$ Amp enhancement $\frac{f/\gamma\omega_0}{f/\omega_0^2} = \frac{\omega_0}{\gamma} = Q$

Res. freq / FWHM $= \omega_0/\sqrt{3}\gamma = Q/\sqrt{3}$