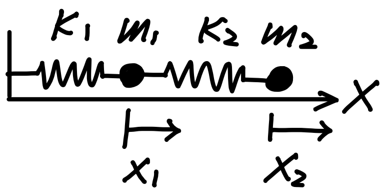
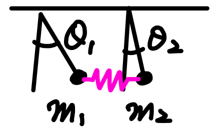
Coupled oscillators

Example 1



Example 2



$f_{BA} = -f_{AB}$

action & reaction forces

$$m_1 x_1'' = -k_1 x_1 - k_2(x_1 - x_2) - b_1 x_1'$$
$$m_2 x_2'' = -k_2(x_2 - x_1) - b_2 x_2'$$

step 1 : organize the terms

$$m_1 x_1'' + b_1 x_1' + (k_1+k_2)x_1 \boxed{-k_2 x_2} = 0$$
$$m_2 x_2'' + b_2 x_2 \boxed{-k_2 x_1} + k_2 x_2 = 0$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}'' + \overbrace{\begin{pmatrix} b_1/m_1 & 0 \\ 0 & b_2/m_2 \end{pmatrix}}^{\equiv \hat{\gamma}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' + \overbrace{\begin{pmatrix} (k_1+k_2)/m_1 & -k_2/m_1 \\ -k_2/m_2 & k_2/m_2 \end{pmatrix}}^{\equiv \hat{M}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

Define $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \vec{x}' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}, \ \vec{x}'' = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}''$

$$\Rightarrow \vec{x}'' + \hat{\gamma}\vec{x}' + \hat{M}\vec{x} = 0$$

Ansatz : $\vec{x} = \vec{A} e^{\alpha t}$, explicitly $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} e^{\alpha t}$

$\vec{x}' = A\alpha e^{\alpha t}$

$\vec{x}'' = A\alpha^2 e^{\alpha t}$

$$\Rightarrow \alpha^2 e^{\alpha t}\vec{A} + \alpha\hat{\gamma}\vec{A}e^{\alpha t} + \hat{M}\vec{A}e^{\alpha t} = 0$$
$$\Rightarrow (\alpha^2 \hat{\mathbb{1}} + \alpha\hat{\gamma} + \hat{M})\vec{A} = 0$$

$\hat{\mathbb{1}}$: identity matrix $\hat{\mathbb{1}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Two options : $\vec{A} = 0 \Rightarrow$ equilibrium

$|\alpha^2 \hat{\mathbb{1}} + \alpha\hat{\gamma} + \hat{M}| = 0$

Given a matrix $\hat{N} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$|\hat{N}| = \det \hat{N} = ad - bc.$

$$\left| \begin{pmatrix} \alpha^2 + \alpha b_1/m_1 + (k_1+k_2)/m_1 & -k_2/m_1 \\ -k_2/m_2 & \alpha^2 + \alpha b_2/m_2 + k_2/m_2 \end{pmatrix} \right| = 0$$

$$\Rightarrow \left(\alpha^2+\frac{b_1}{m_1}\alpha+\frac{k_1+k_2}{m_1}\right)\left(\alpha^2+\frac{b_2}{m_2}\alpha+\frac{k_2}{m_2}\right)-\frac{k_1k_2}{m_1m_2}=0$$

$$\Rightarrow \alpha^4+\cdots\alpha^3+\cdots\alpha^2+\cdots\alpha+\cdots=0$$

$$\Rightarrow \alpha=\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4.$$

For each α_i, we can solve for A_i such that $(\alpha_i^2\hat{\mathbb{I}}+\alpha_i\hat{\gamma}+\hat{M})A_i=0$

$$\Rightarrow \vec{X}=c_1\vec{A}_1e^{\alpha_1 t}+c_2\vec{A}_2e^{\alpha_2 t}+c_3\vec{A}_3e^{\alpha_3 t}+c_4\vec{A}_4e^{\alpha_4 t}$$

Example: $m_1=m_2=m$, $\gamma_1=\gamma_2=0$. $k_1/m_1\equiv\omega$ $k_2/m_2\equiv\omega$

$$\Rightarrow \left[\alpha^2\mathbb{I}+\begin{pmatrix}\omega^2+\omega^2 & -\omega^2\\ -\omega^2 & \omega^2\end{pmatrix}\right]\vec{A}=0 \Rightarrow \begin{pmatrix}2\omega^2 & -\omega^2\\ -\omega^2 & \omega^2\end{pmatrix}\vec{A}=-\alpha^2\vec{A}$$

eigenvalue-eigenvector problem

$$\begin{vmatrix}\alpha^2+2\omega^2 & -\omega^2\\ -\omega^2 & \alpha^2+\omega^2\end{vmatrix}=0$$

$$\Rightarrow (\alpha^2+2\omega^2)(\alpha^2+\omega^2)-\omega^4=0$$

$$\Rightarrow \alpha^4+3\omega^2\alpha^2+\omega^4=0.$$

$$\Rightarrow \alpha^2=-\frac{3}{2}\omega^2\pm\frac{1}{2}\sqrt{(3\omega^2)^2-4\omega^4}$$

$$=-\frac{3}{2}\omega^2\pm\frac{\omega^2}{2}\sqrt{5}$$

$$=\frac{1}{2}(-3\pm\sqrt{5})\omega^2<0 \Rightarrow \alpha=\pm\frac{i}{2}\sqrt{3+\sqrt{5}}\,\omega,\ \pm\frac{i}{2}\sqrt{3-\sqrt{5}}\,\omega$$

For $\alpha_\pm^2=\frac{1}{2}(-3\pm\sqrt{5})\omega^2$: we have $\begin{pmatrix}\alpha_\pm^2+2\omega^2 & -\omega^2\\ -\omega^2 & \alpha_\pm^2+\omega^2\end{pmatrix}\vec{A}_\pm=0$. $\Rightarrow \vec{A}_\pm=\begin{pmatrix}1\\ \frac{1}{2}(1\pm\sqrt{5})\end{pmatrix}$

$$\Rightarrow \vec{X}=c_1\vec{A}_+e^{i\omega_+t}+c_2\vec{A}_+e^{-i\omega_+t}+c_3\vec{A}_-e^{-i\omega_-t}+c_4\vec{A}_-e^{i\omega_+t}$$

$$=\vec{A}_+\left[c_1\cos\omega_+t+c_2\sin\omega_+t\right]+\vec{A}_-\left[c_3\cos\omega_-t+c_4\sin\omega_-t\right]$$

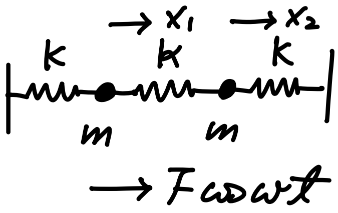
Remark: Four constants for initial positions and velocities of both particles.

$\vec{A}_+$ and $\vec{A}_-$ describe 2 independent modes.

$$\vec{A}_+\cdot\vec{A}_-=1+\frac{1}{4}(1+\sqrt{5})(1-\sqrt{5})=1+\frac{1}{4}(1-5)=0.$$

Symmetric matrix must have orthogonal eigenvectors with diff. eigenvalues.

Driven coupled oscillators



$$m x_1'' = -k x_1 - \kappa(x_1 - x_2) - b x_1' + F\cos\omega t$$
$$m x_2'' = -k x_2 - \kappa(x_2 - x_1) - b x_2'$$

Textbook uses physics intuition to guess the solution. We will do it formally.

$$x_1'' + \gamma x_1' + \frac{k+\kappa}{m}x_1 - \frac{\kappa}{m}x_2 = f e^{i\omega t}$$
$$x_2'' + \gamma x_2' - \frac{\kappa}{m}x_1 + \frac{k+\kappa}{m}x_2 = 0$$

$$\Rightarrow \vec{x}'' + \gamma \vec{x}' + \begin{pmatrix} \frac{k+\kappa}{m} & -\frac{\kappa}{m} \\ -\frac{\kappa}{m} & \frac{k+\kappa}{m} \end{pmatrix}\vec{x} = f\begin{pmatrix}1\\0\end{pmatrix}e^{i\omega t}$$

Approach 1: Ansatz $\vec{x} = \vec{A}e^{i\omega t}$ $\vec{x}' = i\omega\vec{A}e^{i\omega t}$, $\vec{x}'' = -\omega^2\vec{A}e^{i\omega t}$

$$\Rightarrow \left[-\omega^2 + i\gamma\omega + \frac{1}{m}\begin{pmatrix} k+\kappa & -\kappa \\ -\kappa & k+\kappa \end{pmatrix}\right]A = f\begin{pmatrix}1\\0\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \Omega & -\kappa/m \\ -\kappa/m & \Omega \end{pmatrix}A = \begin{pmatrix} f \\ 0 \end{pmatrix} \qquad \Omega = -\omega^2 + i\gamma\omega + (k+\kappa)/m$$

$$-\kappa/m\, A_1 + \Omega A_2 = 0 \Rightarrow A = \begin{pmatrix}\Omega \\ \kappa/m\end{pmatrix} n \Rightarrow (\Omega^2 - \kappa^2/m^2)\, n = f$$

$$\Rightarrow A = \begin{pmatrix}\Omega \\ \kappa/m\end{pmatrix}\frac{f}{\Omega^2 - \kappa^2/m^2}$$

$$\Rightarrow \vec{x} = \begin{pmatrix}\Omega \\ \kappa/m\end{pmatrix} \mathrm{Re}\, \frac{e^{i\omega t}}{\Omega^2 - \kappa^2/m^2}\, f$$

Approach 2: From observation (physics intuition) we can check the center of mass $x_1 + x_2$ and relative $x_1 - x_2$ motion.

$$x_1'' + \gamma x_1' + \frac{k+\kappa}{m}x_1 - \frac{\kappa}{m}x_2 = f e^{i\omega t}$$
$$x_2'' + \gamma x_2' - \frac{\kappa}{m}x_1 + \frac{k+\kappa}{m}x_2 = 0$$

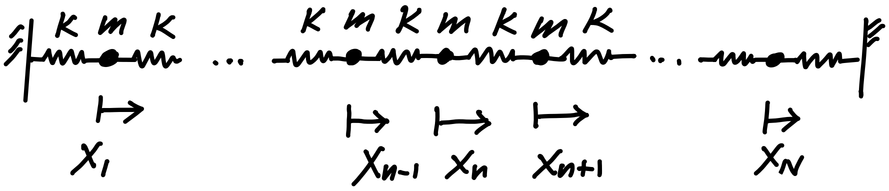
$\Rightarrow$

$X \equiv (x_1 + x_2)/2$
$u \equiv x_1 - x_2$

$$X'' + \gamma X' + \frac{k}{m}X = \frac{f}{2}e^{i\omega t}$$
$$u'' + \gamma u' + \frac{k+2\kappa}{m}u = \frac{f}{2}e^{i\omega t}$$

$\Rightarrow$ Two uncoupled oscillators.

N-coupled oscillators.



x_i : deviation of the ith mass from the equlibrium position.

$$mx_n'' = F_{Left} + F_{Right} = \overbrace{-k(x_n - x_{n-1})}^{F_{left}} \overbrace{-k(x_n - x_{n+1})}^{F_{right}}$$

$$x_n'' = -2kx_n + kx_{n-1} + kx_{n+1}$$

$$\equiv -2\omega_0^2 x_n + \omega_0^2 x_{n-1} + \omega_0^2 x_{n+1}$$

$$= \omega_0^2 (x_{n-1} - 2x_n + x_{n+1})$$

Boundary (imagine $x_0 = x_{N+1} = 0$ because of the walls)

$$x_1'' = \omega_0^2(\quad -2x_1 + x_2)$$
$$x_2'' = \omega_0^2(x_1 - 2x_2 + x_3)$$
$$\vdots$$
$$x_i'' = \omega_0^2(x_{i-1} - 2x_i + x_{i+1})$$
$$\vdots$$
$$x_N'' = \omega_0^2(x_{N-1} - 2x_N \quad)$$

$$\Rightarrow \vec{x}'' = \begin{pmatrix} x_1 \\ x_2 \\ \\ \vdots \\ \\ x_N \end{pmatrix}'' = \omega_0^2 \underbrace{\begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & -2 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & -2 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & -2 & 1 & 0 & \cdots \\ & & \vdots & & \vdots & & \\ 0\,0\,0 & \cdots & & 0 & 1 & -2 & 1 \\ 0\,0\,0 & \cdots & & 0 & 0 & 1 & -2 \end{pmatrix}}_{\hat{M}} \begin{pmatrix} x_1 \\ x_2 \\ \\ \vdots \\ \\ x_N \end{pmatrix}$$

$$\Rightarrow \vec{x}'' = \omega_0^2 \hat{M} \vec{x}$$

Standard approach: Ansatz $\vec{x} = \vec{A} e^{i\omega t} \Rightarrow \vec{x}' = i\omega \vec{A} e^{i\omega t}$, $\vec{x}'' = -\omega^2 \vec{A} e^{i\omega t}$

$\Rightarrow -\omega^2 \vec{A} = \omega_0^2 \hat{M} \vec{A}$

standard eigenvalue-eigenvector problem $\hat{M}\vec{A} = -\frac{\omega^2}{\omega_0^2}\vec{A} \equiv \lambda \vec{A}$

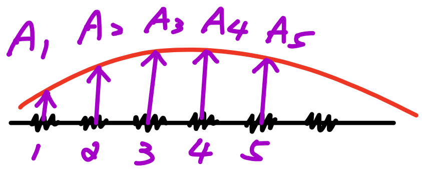
eigenvalue

Math: given a $N \times N$ matrix there are N eigenvalues $2N$ eigenfreq. we need

$2N$ boundary conditions for positions & velocities of all N masses.

Smart guess of the solution

given $-\frac{\omega^2}{\omega_0^2}A_n = A_{n-1} - 2A_n + A_{n+1}$



Ansatz $A_1 = \sin\epsilon$ $\quad A_2 = \sin 2\epsilon$ $\quad A_3 = \sin 3\epsilon$ $\quad A_n = \sin n\epsilon$

$$\Rightarrow \left(2 - \frac{\omega^2}{\omega_0^2}\right)A_n = A_{n-1} + A_{n+1}$$

$$\Rightarrow 2 - \frac{\omega^2}{\omega_0^2} = \frac{A_{n+1} + A_{n-1}}{A_n} = \frac{\sin(n+1)\epsilon + \sin(n-1)\epsilon}{\sin n\epsilon} = \frac{2\sin n\epsilon\cos\epsilon}{\sin n\epsilon} = 2\cos\epsilon$$

$$\Rightarrow \omega^2 = \omega_0^2\, 2(1-\cos\epsilon) = 4\omega_0^2 \sin^2\epsilon/2 \Rightarrow \omega = \pm 2\omega_0 \sin\epsilon/2$$

What are the possible values for ϵ?

From boundary conditions: $A_0 = A_{N+1} = 0$

$\Rightarrow \sin(N+1)\epsilon = 0$ or $(N+1)\epsilon = m\pi$, $\epsilon_m = m\pi/(N+1)$, $\quad m = \cancel{0}, 1, 2, 3 \cdots N, \cancel{N+1}$

both lead to zero displacement

$\Rightarrow \omega_m = \pm 2\omega_0 \sin\epsilon_m/2$.

Sanity check: N eigenmodes, $2N$ eigenfreqs for N particles

General solution $(A_0 = A_{N+1} = 0)$

$$X_n(t) = \sum_{m=1}^{N} C_m \sin n\epsilon_m e^{-i\omega_m t} + D_m \sin n\epsilon_m e^{i\omega_m t}, \qquad \epsilon_m = m\pi/(N+1)$$

$$= \sum_m \sin n\epsilon_m (E_m \cos\omega_m t + F_m \sin\omega_m t) \qquad \omega_m = 2\omega_0 \sin\epsilon_m$$

Without the B.C show that the general solution is

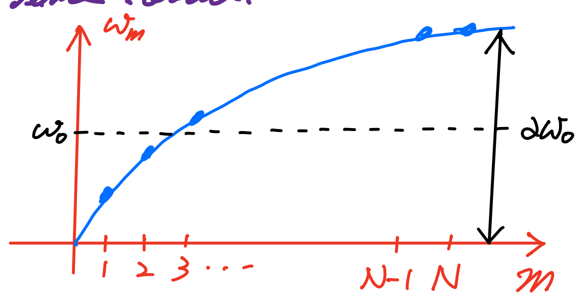
$$X_n(t) = \sum_{m=1}^{N} (G_m \cos n\epsilon_m + H_m \sin n\epsilon_m)(I_m \cos\omega_m t + J_m \sin\omega_m t)$$

$$= \sum_{m=1} K_m e^{i(n\epsilon_m + \omega_m t)} + L_m e^{i(-n\epsilon_m + \omega_m t)} + M_m e^{i(n\epsilon_m - \omega_m t)} + N_m e^{-i(n\epsilon_m + \omega_m t)}$$

Check textbook ansatz $A_n = \cos n\epsilon$, which gives the same result.

Complex ansatz $A_n = e^{in\epsilon}$

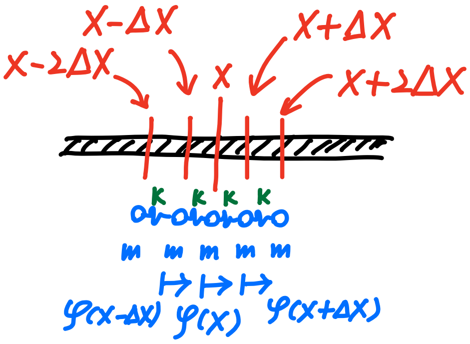
$$\Rightarrow 2 - \frac{\omega^2}{\omega_0^2} = \frac{e^{i(n+1)\epsilon} + e^{i(n-1)\epsilon}}{e^{in\epsilon}} = e^{i\epsilon} + e^{-i\epsilon} = 2\cos\epsilon$$



Continuous mass-spring system: Wave eqn.

We model an elastic string as an infinite mass-spring chain

rubber band



linear density

Each section Δx has mass $m = \rho \Delta x$

k: force constant between 2 sections

$\varphi(x)$ displacement of the section @ x

$$m\, \partial_t^2 \varphi(x,t) = k\left[\varphi(x-\Delta x) - 2\varphi(x) + \varphi(x+\Delta x)\right]$$

$$\rho \Delta x\, \partial_t^2 \varphi(x,t) = k\left[\frac{\varphi(x+\Delta x)-\varphi(x)}{\Delta x} - \frac{\varphi(x)-\varphi(x-\Delta x)}{\Delta x}\right]\Delta x$$

$$= k\left[\partial_x \varphi(x+\tfrac{\Delta x}{2}) - \partial_x \varphi(x-\tfrac{\Delta x}{2})\right]\Delta x = k\, \partial_x^2 \varphi(x)\, \Delta x^2$$

$$\rho\, \partial_t^2 \varphi = k\Delta x\, \partial_x^2 \varphi \equiv \varepsilon\, \partial_x^2 \varphi \qquad \varepsilon: \text{Young's modulus.}$$

Wave equation $\partial_t^2 \varphi = v^2 \partial_x^2 \varphi$ 2nd order partial diff. eqn. $v^2 = \varepsilon/\rho$

Linearity: If φ_1 & φ_2 are solutions, so is $A\varphi_1 + B\varphi_2$ a solution.

Ansatz $\varphi(x,t) = A(x)e^{i\omega t}$

$$\Rightarrow -\omega^2 A \cancel{e^{i\omega t}} = v^2 A'' \cancel{e^{i\omega t}}$$

$$\Rightarrow A'' = -\frac{\omega^2}{v^2} A$$

Use ansatz again $A(x) \equiv e^{ikx}$. $A'' = -k^2 A$

$$\Rightarrow k = \pm \frac{\omega}{v}$$

$$-\omega^2 \cancel{e^{ikx}} = -v^2 k^2 \cancel{e^{ikx}}$$

$$\Rightarrow \omega^2 = v^2 k^2 \Rightarrow \omega = \pm v k$$

$$\Rightarrow \varphi = (Ae^{ikx} + Be^{-ikx})(Ce^{i\omega t} + De^{-i\omega t})$$

$$= A'e^{i(kx+\omega t)} + B'e^{i(kx-\omega t)} + C'e^{i(-kx+\omega t)} + D'e^{i(-kx-\omega t)}$$

$$= A'e^{ik(x+vt)} + B'e^{ik(x-vt)} + C'e^{-ik(x-vt)} + D'e^{-ik(x+vt)}$$

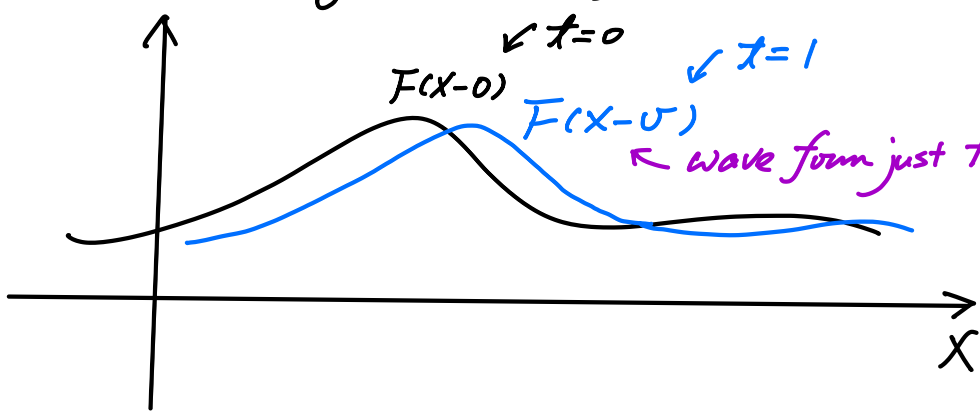
In fact any function $F(\underbrace{x \pm vt}_{u})$ are solutions of the wave eqn.

Proof: Left side: $\partial_x F = \frac{\partial}{\partial x} F(u) = \frac{\partial u}{\partial x}\frac{dF}{du} = F'$, $\partial_x^2 F = F''$

Right side: $\partial_t F = \frac{\partial u}{\partial t}\frac{dF}{du} = \pm v F'$, $\partial_t^2 F = v^2 F''$

$F(x-vt)$ is a travelling wave moving in the $+x$ direction

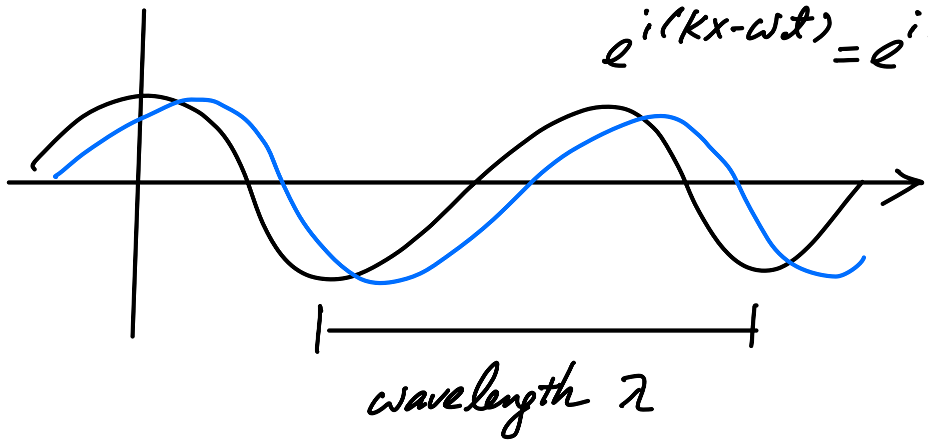
$F(x+vt)$ is a travelling wave moving in the $-x$ direction



← wave form just translates by v per unit time.

For a sinusoidal wave

$e^{i(kx-\omega t)} = e^{ik(x-vt)}$



wavenumber $k = 2\pi/\lambda$

angular freq $\omega = kv$

wave velocity $v = \frac{\omega}{k} = f\lambda$