Fourier expansion and Fourier transform

How do we solve $\hat{\mathcal{L}} x = f$ with arbitrary force f ?

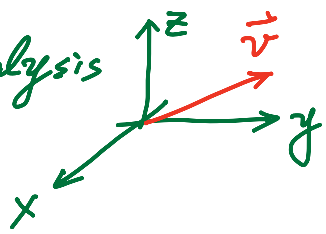
- single particle : $x'' + \gamma x' + \omega_0^2 x = f(t)$
- multiparticles : $\vec{x}'' + \hat{\gamma}\vec{x}' + \hat{M}\vec{x} = \vec{f}(t)$
- Continuous system $\partial_t^2 \varphi - v^2 \partial_x^2 \varphi = f(x,t)$

Claim : Once you know how to solve $f(t) = e^{\alpha t}$, you know how to solve any function $f(t)$

Idea:
$$\begin{aligned} \hat{\mathcal{L}} x_1 &= A_1 e^{\alpha_1 t} \\ \hat{\mathcal{L}} x_2 &= A_2 e^{\alpha_2 t} \\ &\vdots \\ + \quad & \\ \hline \hat{\mathcal{L}} \textstyle\sum_i x_i &= \textstyle\sum_i A_i e^{\alpha_i t} \equiv f(t) \end{aligned}$$

Math : Any continuous function $y(x)$ can be expanded by a set of complete set of $\cos\alpha x$, $\sin\alpha x$ or e^{ikx}.

vector analysis

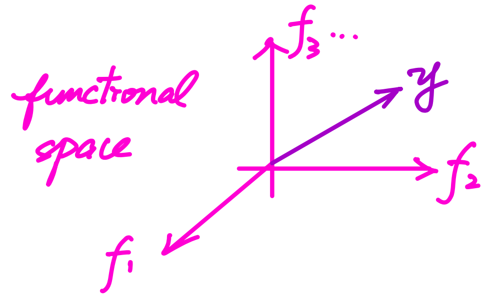


any vector in 3D can be $\vec{v} = v_x \hat{x} + v_y \hat{y} + v_z \hat{z}$

$\hat{x}, \hat{y}, \hat{z}$ form a complete and orthogonal set of basis.

$\hat{x}\cdot\hat{x} = \hat{y}\cdot\hat{y} = \hat{z}\cdot\hat{z} = 1$ $\quad$ $\vec{v} = v_x \hat{x} + v_y \hat{y} + v_z \hat{z}$

$\hat{x}\cdot\hat{y} = \hat{y}\cdot\hat{z} = \hat{x}\cdot\hat{z} = 0$. $\quad$ $v_x = \vec{v}\cdot\hat{x}$, $v_y = \vec{v}\cdot\hat{y}$, $v_z = \vec{v}\cdot\hat{z}$



$\{f_i\}$ form a complete and orthogonal set of basis.

scalar product in functional space.

Orthonormal basis: $\boxed{\int f_i^*(x) f_j(x)\, dx} = \delta_{ij}$. $\quad \delta_{ij} = 1$ when $i=j$
$\qquad = 0$ when $i \neq j$

$y(x) = \Sigma_i \underbrace{A_i}_{\text{ampu.}} \underbrace{f_i(x)}_{\text{basis}}$. $\quad$ Amplitude $A_i = \int f_i^*(x) y(x)\, dx$

Fourier Expansion for a periodic function $y(x+L) = y(x)$

Claim: $y(x) = A_0 + \Sigma_n A_n \cos n\frac{2\pi x}{L} + \Sigma_n B_n \sin n\frac{2\pi x}{L}$

The complete set here is $\{1.\ \cos n\frac{2\pi x}{L}.\ \sin n\frac{2\pi x}{L}\}$

These are all sin & cos with same period L.

Check orthogonality: $\int_0^L 1\cdot\cos n\frac{2\pi x}{L}\,dx = \int_0^L 1\cdot\sin n\frac{2\pi x}{L}\,dx = \int_0^L \sin n\frac{2\pi x}{L}\cos m\frac{2\pi x}{L}\,dx = 0$

$\int_0^L 1\cdot 1\,dx = L$. $\int_0^L \sin n\frac{2\pi x}{L}\sin m\frac{2\pi x}{L}\,dx = \int_0^L \cos n\frac{2\pi x}{L}\cos n\frac{2\pi x}{L}\,dx = \frac{L}{2}\delta_{mn}$

$$\Rightarrow y(x) = \boxed{\frac{1}{L}\int_0^L y(x)\,dx} + \boxed{\frac{2}{L}\Sigma_n \cos n\frac{2\pi x}{L}\int_0^L \cos n\frac{2\pi x}{L}\, y(x)\,dx}$$

cos terms

constant term

$$+ \boxed{\frac{2}{L}\Sigma_n \sin n\frac{2\pi x}{L}\int_0^L \sin n\frac{2\pi x}{L}\, y(x)\,dx}$$

sin terms

Remark: The orthonormal basis is $\{\underline{\frac{1}{\sqrt{L}}}.\ \underline{\sqrt{\frac{2}{L}}\cos n\frac{2\pi x}{L}}.\ \underline{\sqrt{\frac{2}{L}}\sin n\frac{2\pi x}{L}}\}$

constant term is just the mean.

Alternatively, we can use exp function

$$y(x) = A_0 + \sum_n A_n \left(e^{-in2\pi x/L} + e^{in2\pi x/L}\right) + \sum_n B_n \left(e^{-in2\pi x/L} - e^{in2\pi x/L}\right)$$

$$= \sum_{n=-\infty}^{\infty} C_n \underbrace{e^{in2\pi x/L}}_{\text{basis}}$$

$$\int_0^L e^{in2\pi x/L} e^{-im2\pi x/L} dx = \frac{1}{L}\delta_{nm}$$

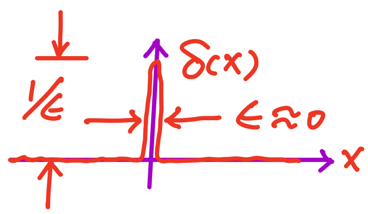
Fourier transform. (general case of $y(x)$)

Now let $L \to \infty$.

$$y(x) = \int_{-\infty}^{\infty} \underbrace{y(k)}_{\text{coeff.}} \underbrace{e^{ikx}}_{\text{basis}} dk .$$

$$\int_{-\infty}^{\infty} e^{-ik'x} e^{ikx} dx = 2\pi \delta(k-k')$$

Dirac's δ function: Unity area but infinitely narrow.



$\delta(x) = 0$ for $x \neq 0$

$\delta(x) \to \infty$ for $x = 0$

$\int \delta(x) dx = 1$

$$\int f(x)\delta(x) dx = f(0).$$

$$\int e^{-ikx} y(x) dx$$
$$= \iint e^{-ikx} y(k') e^{ik'x} dk' dx$$
$$= \int y(k') \int e^{i(k'-k)x} dx\, dk'$$
$$= 2\pi \int y(k') \delta(k'-k) dk'$$
$$= 2\pi y(k)$$
$$\Rightarrow y(k) = \frac{1}{2\pi} \int y(x) e^{-ikx} dx$$

$$\begin{cases} y(x) = \int y(k) e^{ikx} dk & \leftarrow \text{Fourier transform} \\ y(k) = \frac{1}{2\pi} \int y(x) e^{-ikx} dx & \leftarrow \text{inverse Fourier transform} \end{cases}$$

$$\begin{cases} y(t) = \int y(\omega) e^{i\omega t} d\omega & \leftarrow \text{Fourier transform} \\ y(\omega) = \frac{1}{2\pi} \int y(t) e^{-i\omega t} dt & \leftarrow \text{inverse Fourier transform} \end{cases}$$

$y(\omega)$: temporal spectral function in freq domain

$y(t)$: temporal function in time domain

$y(k)$: momentum space

$y(x)$: real space.