Solving ODE $a(x)y''(x)+b(x)y'(x)+c(x)y(x)=f(x)$

Identify the homogeneous solutions $u_1(x).\ u_2(x)$ s.t.

$$\hat{\mathcal{L}}u_i \equiv [a(x)d_x^2+b(x)d_x+c(x)]u_i(x)=0$$

Ansatz $y=v_1(x)u_1(x)+v_2(x)u_2(x)$

$$y'=v_1(x)u_1'(x)+v_2(x)u_2'(x)+u_1(x)v_1'(x)+u_2(x)v_2'(x)$$

$$y''=v_1(x)u_1''(x)+v_2(x)u_2''(x)+u_1'(x)v_1'(x)+u_2'(x)v_2'(x)+\frac{d}{dx}(u_1v_1'+u_2v_2')$$

If we demand $\begin{cases} u_1v_1'+u_2v_2'=0 \\ u_1'v_1'+u_2'v_2'=f/a \end{cases}$, we have

$$\hat{\mathcal{L}}y=ay''+by'+cy=v_1\cancelto{0}{\hat{\mathcal{L}}u_1}+v_2\cancelto{0}{\hat{\mathcal{L}}u_2}+a(u_1'v_1'+u_2'v_2')=f$$

Thus $y=v_1u_1+v_2u_2$ is a solution, v_1 and v_2 satisfy

$$\begin{pmatrix} u_1 & u_2 \\ u_1' & u_2' \end{pmatrix}\begin{pmatrix} v_1' \\ v_2' \end{pmatrix}=\begin{pmatrix} 0 \\ f/a \end{pmatrix} \Rightarrow \begin{array}{l} u_1'u_1v_1'+u_1'u_2v_2'=0 \\ u_1'u_1v_1'+u_1u_2'v_2'=u_1f/a \end{array}$$

$$\Rightarrow v_2'=\frac{u_1f/a}{u_1u_2'-u_1'u_2}$$

$$\Rightarrow v_1'=\frac{-u_2f/a}{u_1u_2'-u_1'u_2}$$

Define $W(x)=\left|\begin{pmatrix} u_1 & u_2 \\ u_1' & u_2' \end{pmatrix}\right|=u_1u_2'-u_1'u_2$

$$\Rightarrow y=-u_1\int\frac{u_2f}{aW}dx+u_2\int\frac{u_1f}{aW}dx$$