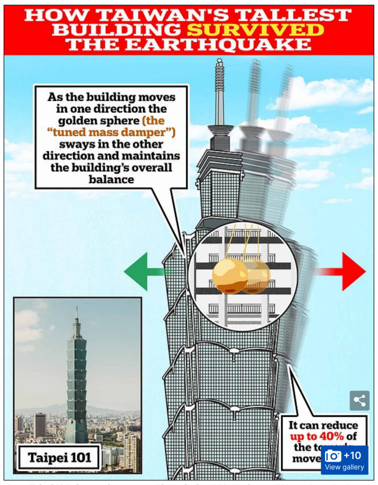


Key to Taipei 101's impressive structural integrity is a 660 metric ton golden sphere that hangs from the 92nd floor

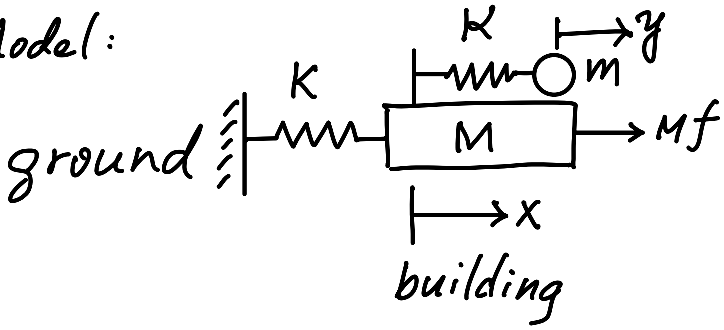
Moment 508-metre-high Taipei 101 building sways during earthquake

Tuned Mass Damper used in Taipei 101

Idea: Dissipate earthquake induced vibrations by a damped pendulum.

M: building mass
m: pendulum mass
$\omega_0^2 = K/M$ building natural freq.
$\omega_1^2 = \kappa/m$ pendulum nat. freq.
γ_1: building damping
γ_2: pendulum damping
Mf: earthquake force

Model:



Eq. of motion

$$\begin{cases} M x'' = -Kx - \kappa(x-y) + \beta_1 y' - \beta x' + M f e^{i\omega t} \\ m y'' = -\kappa(y-x) - \beta_1 y' \end{cases}$$

$\gamma_0 \equiv \beta/M$, $\omega_0^2 \equiv K/M$
$\gamma_1 \equiv \beta_1/m$ $\omega_1^2 \equiv \kappa/m$
$\kappa/M = \omega_1^2\, m/M$
$\beta_1/M = \gamma_1\, m/M$

$$\Rightarrow \quad \begin{aligned} & x'' + \gamma_0 x' + \omega_0^2 x + \frac{\kappa}{M}(x-y) - \frac{\beta_1}{M} y' = f e^{i\omega t} \\ & y'' + \gamma_1 y' - \omega_1^2 x + \omega_1^2 y = 0 \end{aligned}$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix}'' + \begin{pmatrix} \gamma_0 & -\gamma_1 m/M \\ 0 & \gamma_1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}' + \begin{pmatrix} \omega_0^2 + \frac{m}{M}\omega_1^2 & -\frac{m}{M}\omega_1^2 \\ -\omega_1^2 & \omega_1^2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix} e^{i\omega t}$$

Ansatz: $\begin{pmatrix} x \\ y \end{pmatrix} = \vec{A} e^{i\omega t} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} e^{i\omega t} \Rightarrow (-\omega^2 + i\omega \hat{\gamma} + \hat{M}) \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \equiv \begin{pmatrix} f \\ 0 \end{pmatrix}$

$$\Rightarrow \left[\begin{pmatrix} -\omega^2 & 0 \\ 0 & -\omega^2 \end{pmatrix} + \begin{pmatrix} i\omega\gamma_0 & -i\omega\gamma_1 m/M \\ 0 & i\omega\gamma_1 \end{pmatrix} + \begin{pmatrix} \omega_0^2 + \frac{m}{M}\omega_1^2 & -\frac{m}{M}\omega_1^2 \\ -\omega_1^2 & \omega_1^2 \end{pmatrix} \right] \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

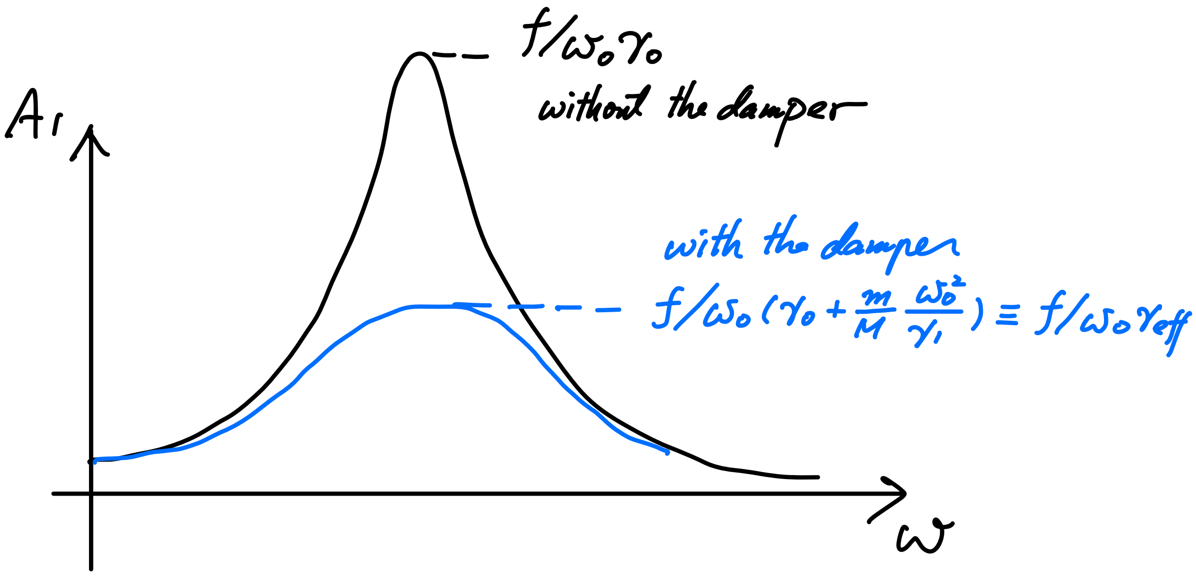
$$\Rightarrow \begin{bmatrix} \omega_0^2-\omega^2+\frac{m}{M}\omega_1^2+i\omega\gamma_0 & -i\omega\gamma_1 m/M - \omega_1^2 m/M \\ -\omega_1^2 & \omega_1^2-\omega^2+i\omega\gamma_1 \end{bmatrix}\begin{pmatrix}A_1\\A_2\end{pmatrix}=\begin{pmatrix}f\\0\end{pmatrix}$$

From 2nd Eq. $\Rightarrow (\omega_1^2-\omega^2+i\omega\gamma_1)A_2 = A_1\omega_1^2$

$$\Rightarrow \left(\omega_0^2-\omega^2+\frac{m}{M}\omega_1^2+i\omega\gamma_0\right)A_1 - \left(i\omega\gamma_1 m/M+\omega_1^2 m/M\right)\frac{\omega_1^2 A_1}{\omega_1^2-\omega^2+i\omega\gamma_1} = f$$

$$\Rightarrow A_1 = \frac{f}{\omega_0^2-\omega^2+\frac{m}{M}\omega_1^2+i\omega\gamma_0-\frac{m\omega_1^2}{M}\frac{\omega_1^2+i\omega\gamma_1}{\omega_1^2-\omega^2+i\omega\gamma_1}}$$

Near resonance $\omega\approx\omega_0\approx\omega_1$ $\quad A_1 = \dfrac{f}{i\omega_0\gamma_0+i\frac{m}{M}\frac{\omega_0^3}{\gamma_1}} \Rightarrow |A_1| = \dfrac{f}{\omega_0\gamma_0+\frac{m}{M}\frac{\omega_0^3}{\gamma_1}}$



Effective new damping $\gamma_{eff} = \gamma_0 + \frac{m}{M}\frac{\omega_0^2}{\gamma_1}$

Building shaking amp. is reduced by a factor of $\left(1+\frac{m}{M}\frac{\omega_0^2}{\gamma_0\gamma_1}\right)$.