Physics 471 – Introduction to Modern Atomic Physics

HOMEWORK I (Due: 1/15/2024)

1. Stable atom isotopes

Let's analyze the 20 lightest element from hydrogen to calcium.

a. Identify all stable isotopes and list their atomic number Z and neutron number N=mass number A - Z.

b. What is the fraction of isotopes with even Z? And the fraction with even N?

c. What caused the different abundance of isotopes with even and odd Z/N? Provide a citation.

d. If an atom is composed of even fermions (electrons+protons+neutrons), it is a boson. Otherwise, a fermion. Is it more likely that an atom is boson or fermion?

e. Can you identify other patterns or rules of isotope stability?

2. Fine structure an atom

Fine structure of an atom comes mostly from the coupling between the orbital angular momentum of the electrons and their spins. The interaction Hamiltonian can be approximated by

$$V_F = \gamma\, \vec{L} \cdot \vec{S},$$

where $\vec{L}$ is the orbital angular momentum, $\vec{S}$ is the spin and $\vec{J} = \vec{L} + \vec{S}$ is the total angular momentum, and the constant γ characterizes the strength of the fine structure interactions. constant.

a. Determine the fine structure energy of an electronic orbital $^{2S+1}L_J$
(Hint: Diagonalize the Hamiltonian given S, L and J.)

b. For alkali atoms in the first excited state $ns\,^2P_J$ with $L = 1$ and $S = 1/2$, determine the energy splitting between the levels $J = 3/2$ and $J = 1/2$ in terms of γ.

c. Look up online the fine structure splitting of H, Li, Na, K, Rb, Cs and Fr in their first excited $L = 1$ state. Test textbook's claim that the fine structure coupling γ scales with the atomic number Z as $\gamma \propto Z^2$.
(Hint: you can find the data here: https://steck.us/alkalidata/)

d. D1 and D2 lines of alkali-like atoms refers to the optical transitions between the ground state with $L = 0$ and the first excited states with $L = 1$ and $J = \frac{1}{2}, \frac{3}{2}$, respectively. As an example, the D1 and D2 line wavelengths are 795 nm and 780 nm for Rubidium. Predict the D1 and D2 line wavelengths of Ununennium ^{119}Uue.

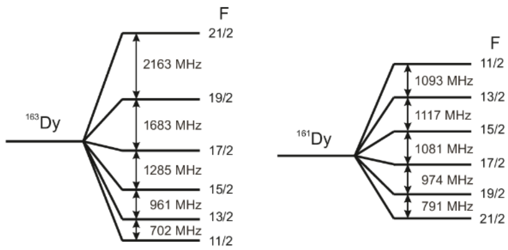
3. Hyperfine structure an atom

Hyperfine structure of an atom comes from the coupling between the total angular momentum of the electrons $\vec{J}$ and the nuclear spin $\vec{I}$. The hyperfine interaction Hamiltonian is very well approximated by

$$V_{HF} = \eta\, \vec{J} \cdot \vec{I},$$

which has the same form as the fine structure interactions. The total angular momentum is $\vec{F} = \vec{J} + \vec{I} = \vec{L} + \vec{S} + \vec{I}$.

Dysprosium has a complex hyperfine structure, shown below. Determine J, I and η for the two isotopes. [ ]



4. Atom in a magnetic field: Breit-Rabi formula

In the presence of an external magnetic field $\vec{B}$, the Hamiltonian of an alkali atom in the ground state with $L = 0, S = \frac{1}{2}$, and nuclear spin I is given by

$$H = \gamma\, \vec{S} \cdot \vec{I} - \vec{\mu}_S \cdot \vec{B} - \vec{\mu}_I \cdot \vec{B},$$

where $\vec{\mu}_S = -g_S \mu_B \vec{S}/\hbar$ is the magnetic moment of the electron, $\vec{\mu}_I = g_I \mu_N \vec{I}/\hbar$ is the magnetic moment of the nucleus, gs are the g-factors, μ_B and μ_N are the Bohr and nucleus magneton.

Here we will show that the eigen-energies are described by the Breit-Rabi formula

$$E = -\frac{E_{HF}}{2(2I+1)} \pm \frac{E_{HF}}{2}\sqrt{1 + \frac{4m_F}{2I+1}x + x^2} - g_I \mu_N m_F B,$$

where $E_{HF} = \left(I + \frac{1}{2}\right)\gamma\hbar^2$ is the hyperfine splitting at zero magnetic field, $m_F = m_I + \frac{1}{2}$, and $x = \frac{g_S \mu_B + g_I \mu_N}{E_{HF}}$ B.

a. Writing $\vec{S} \cdot \vec{I} = \frac{1}{2}(S_+ I_- + S_- I_+) + S_z I_z$, show that the Hamitonian only couples $|m_I, m_S = \frac{1}{2}>$ and $|m_I + 1, m_S = -\frac{1}{2}>$. In this sub-space, the Hamiltonian can be written as

$$H = \begin{bmatrix} \frac{E_{HF}}{2I+1} m_I + \frac{g_S \mu_B B}{2} - g_I \mu_N m_I B & \frac{E_{HF}}{2I+1}\sqrt{I(I+1) - m_I(m_I+1)} \\ \frac{E_{HF}}{2I+1}\sqrt{I(I+1) - m_I(m_I+1)} & -\frac{E_{HF}}{2I+1}(m_I+1) - \frac{g_S \mu_B B}{2} - g_I \mu_N (m_I+1) B \end{bmatrix}$$

b. Diagonalize the matrix and show that you get the Breit-Rabi formula.

c. There are two states in the basis of $|m_I, m_S>$ that do not mix with other states and scale linearly with the magnetic field. Show that in these two states, the electron and nucleus spins are aligned with the field.