Physics 471 – Introduction to Modern Atomic Physics

HOMEWORK 2 (Due: 1/26/2024)

1. Photo scattering

Two useful formulas that describe the interactions between photons and an atom are

Photon scattering rate $s = \frac{\Gamma}{2}\frac{\frac{I}{I_S}}{1+\frac{I}{I_S}+\frac{4\Delta^2}{\Gamma^2}}$ describes how many photons are scattered by an atom.

Light shift $\Delta E = \frac{\Gamma}{8}\frac{\frac{I}{I_S}}{\Delta/\Gamma}$ describes how much the ground state of the atom is shifted by light.

Here I is the intensity of the laser, I_{sat} is the saturation intensity of the atomic transition, Γ is the atomic linewidth, $\Delta = \omega - \omega_0$ is the laser detuning, ω is the laser angular frequency, ω_0 is the atomic transition frequency. We will drive the first formula here.

Starting from the Hamiltonian under rotating wave approximation
$H = \begin{pmatrix} \hbar\Delta/2 & \hbar\Omega/2 \\ \hbar\Omega/2 & -\hbar\Delta/2 \end{pmatrix} = -\frac{\hbar\Delta}{2}\sigma_z + \frac{\hbar\Omega}{2}\sigma_x$, where $\vec{\sigma} = [\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}]$ is the Pauli matrix, Ω is the optical Rabi frequency, and the wavefunction $|\psi >= \psi_1|e > +\psi_2|g >= \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$, show that the density matrix $\rho = |\psi >< \psi|$ follows the *optical Bloch equations*: (ignore the terms in red for now):

$$\partial_t\rho_{11} = i\Omega(\rho_{12} - \rho_{21}) - \Gamma\rho_{11}$$
$$\partial_t\rho_{22} = -i\Omega(\rho_{12} - \rho_{21}) + \Gamma\rho_{11}$$
$$\partial_t\rho_{12} = i\Omega(\rho_{22} - \rho_{11}) + i\Delta\rho_{12} - \frac{\Gamma}{2}\rho_{12}$$
$$\partial_t\rho_{21} = -i\Omega(\rho_{22} - \rho_{11}) - i\Delta\rho_{21} - \frac{\Gamma}{2}\rho_{21},$$

where $\rho_{11} = |\psi_1|^2$ is the excited state population, $\rho_{22} = |\psi_2|^2 = 1 - \rho_{11}$ is the ground state population and $\rho_{12} = \psi_2^*\psi_1 = \rho_{21}^*$ shows the coherence between the states.

A. Rabi flopping occurs when the laser excites an atom. Assume the atom is initially in the ground state $\rho_{22}(0) = 1$. Show that the population oscillates at the generalized Rabi frequency $\Omega_R = \sqrt{\Omega^2 + \Delta^2}$. On resonance, the oscillation frequency is exactly the Rabi frequency Ω.

B. Including the phenomenological damping term Γ, which describes how fast an excited atom decay in vacuum. In the presence of a laser, show that the excited state population $\rho_{11}(t)$ settles to a constant after a long time. Derive the scattering rate using $s \equiv \Gamma\rho_{11}$ and $\frac{I}{I_S} \equiv \frac{2\Omega^2}{\Gamma^2}$.

C. Outline the derivation of the light shift formula based on the optical Bloch equations.

2. Laser cooling and trapping

Here we will derive the cooling and trapping force from a magneto-optical trap. Assume an atom located at x with velocity v is illuminated by counter propagating laser beams red detuned from the atomic transition with detuning $\Delta = \omega - \omega_0 < 0$. A pair of coils introduces a magnetic field gradient on the atoms.

Consider the atomic motion in one dimension and two lasers are propagating along $\pm x$ −direction. Magnetic field is given by $B(x) = b'x$ with a constant gradient b' =const.

A. Show that due to Doppler shift $\Delta_D = -\frac{v}{c}\omega_0$ of the lasers along the $\pm x$ −direction and the Zeeman shift $\Delta_B = -g\mu_B b'x$ the effective detunings of the lasers are $\Delta_\pm = \Delta \pm \Delta_D \pm \Delta_B$. Thus the associated photon scattering rates are $s_\pm = \Gamma\frac{I/I_{sat}}{1+4\Delta_\pm^2/\Gamma^2}$. (We assume $I \ll I_{sat}$.) Show that the total force on the atoms is $f = \hbar k(s_- - s_+)$. To leading order in the atomic velocity v and position x, show that the atomic motion is described by a damped oscillator as

$$\ddot{x} + 2\beta\dot{x} + \gamma x = 0,$$

where the damping coefficient $\beta = 8\Gamma\frac{\hbar}{m}\frac{I}{I_{sat}}\frac{\Delta^2/\Gamma^2}{\left(1+\frac{4\Delta^2}{\Gamma^2}\right)^2}$ determines the cooling rate and $\gamma = 8\Gamma\frac{\hbar k}{m}\frac{I}{I_{sat}}\frac{\Delta^2/\Gamma^2}{\left(1+\frac{4\Delta^2}{\Gamma^2}\right)^2}b'$ the trapping strength.

B. Under this approximation, show that the damping is maximized $\beta_{max} = \frac{\Gamma}{2}\frac{\hbar}{m}\frac{I}{I_{sat}}$ when the detuning is $\Delta = -\frac{\Gamma}{2}$.

3. Bloch vector and Bloch sphere

Given a two-level atom |0> and |1>, its quantum state can be described by the wavefunction $|\psi >= \psi_0|0> + \psi_1|1> \equiv \begin{pmatrix}\psi_0\\ \psi_1\end{pmatrix}$, which satisfies Schroedinger's equation $i\hbar\partial_t|\psi(t)> = H|\psi(t)>$. The wavefunction rotates in the Hilbert space as $|\psi(t)> = U(t)|\psi(0)>$, where $U(t) = e^{-iHt/\hbar}$ is the unitary evolution operator.

The Hamiltonian of the 2-level atom can be written in terms of a 2x2 matrix as $H = \frac{\hbar}{2}(\omega_0\hat{1} + \vec{\omega}\cdot\vec{\sigma})$. The quantum state of the atom can be fully described by the Bloch vector $\vec{b} \equiv < \psi(t)|\vec{\sigma}|\psi(t)>$, where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the Pauli matrices and $\hat{1}$ is the identify matrix. Here $\vec{b}$ and $\vec{\omega}$ are both 3-dimensional vectors with real elements that parameterize the quantum state and the Hamiltonian.

A. Show that as time evolves the Bloch vector rotates around $\vec{\omega}$ according to the Bloch equation $\partial_t\vec{b} = \vec{\omega}\times\vec{b}$.

B. When the atom is coupled to a near-resonant radiation field, we have $\vec{\omega} = (Re[\Omega], Im[\Omega], -\Delta)$, where Ω is the complex Rabi frequency and $\Delta = \omega - \omega_0$ is the laser detuning. (You may use the result from problem #1.)

Describe the motion of a generic Bloch vector when (1) laser is off Ω=0 (2) laser is on resonance $\Delta = 0$ and the electric field is $E = E_0\cos\omega_0 t$ (Ω is real) and (3) laser is on resonance $\Delta = 0$ and the electric field $E = E_0\sin\omega_0 t$ (Ω is imaginary).

C. In quantum information one applies pulses to qubits to manipulate their quantum states. Given the time evolution operator $U(t) = e^{-\frac{iHt}{\hbar}}$, we may define a ϕ *-pulse* along the j -axis as $\phi_j \equiv U(\omega_k t = \phi\delta_{jk})$, where $j, k = x, y, z$. Determine the evolution operator in the matrix form of (a) a π-pulse along the *x*-axis (defined as π_x), (b) a $\frac{\pi}{2}$-pulse along the *y*-axis (defined as $\pi/2_y$), and (c) a 2π-pulse along any direction.

D. Ramsey spectroscopy is widely used in precision metrology. In the simplest form it can be written as

$$\psi_i \longrightarrow \boxed{\pi/2_x} \longrightarrow \boxed{\theta_z} \longrightarrow \boxed{\pi/2_y} \longrightarrow \psi_f$$

Here we start with an atom in the ground state $|\psi_i> = |\psi(t=0)> = \begin{pmatrix}0\\ 1\end{pmatrix}$. After applying a $\frac{\pi}{2}$-pulse along the x -direction, we hold the system for time T, during which the system rotates in the *z*-direction with $\theta = (\omega - \omega_0)T$. We then apply the second $\frac{\pi}{2}$-pulse along the y -direction. Use the result in C. and determine the final state $|\psi_f>$. Show that the excited state population is extremely sensitive to the detuning for a long hold time T.