

HOMEWORK 3

(Due: 2/5/2024)

1. Jayne-Cummings model

A fully quantum mechanical description of an atom interacting with a photon field is given by the *Jayne-Cummings model*

$$H = \frac{\hbar\omega_0}{2}\sigma_z + \hbar\omega b^\dagger b + g(\sigma^+ b + \sigma^- b^\dagger),$$

where $\hbar\omega_0$ is the atomic resonance frequency, ω is the laser frequency, $b^\dagger(b)$ is the creation (annihilation) operator of the photon field, and g is the atom-photon coupling constant, $\vec{\sigma}$ is the Pauli matrix, and $\sigma^\pm = (\sigma_x \pm i\sigma_y)/2$ is the raising/lowering operator of the atomic excitation. The wavefunction of the system combines the two-level atom and photon as $|\Psi\rangle = |e/g\rangle \otimes |n\rangle$, where $n = 0,1,2 \dots$ is the number of photons in the field.

A. First we compare this Hamiltonian with the semi-classical Hamiltonian $H_{semi} = \frac{\hbar}{2}(\Omega\sigma_x - \Delta\sigma_z)$, where $\Delta = \omega - \omega_0$ is the laser detuning. Evaluate the Hamiltonian in the matrix form in the bases of $|e, n\rangle$ and $|g, n + 1\rangle$:

$$H = \begin{pmatrix} \langle e, n | H | e, n \rangle & \langle e, n | H | g, n + 1 \rangle \\ \langle g, n + 1 | H | e, n \rangle & \langle g, n + 1 | H | g, n + 1 \rangle \end{pmatrix}$$

and shows that it is equivalent to the semi-classical Hamiltonian in the basis of $|e\rangle$ and $|g\rangle$ when the photon number is large $n \gg 1$. More explicitly, show that

$$H \approx H_{semi} \otimes |n\rangle + const.,$$

When we identify the Rabi frequency in terms of the coupling constant as $\hbar\Omega = g\sqrt{n}$. The scaling on \sqrt{n} can be understood as the photon number is proportional to the intensity and thus the electric field squared $n \propto I \propto |E|^2$.

B. For a laser field with positive detuning $\Delta = \omega - \omega_0 > 0$ and $\Delta \ll \omega, \omega_0$, calculate the eigenenergies of the lowest 5 states. As an example, the eigenenergies of the Hamiltonian in the lowest 3 states can be evaluated from diagonalizing the following Hamiltonian:

$$H = \begin{pmatrix} \langle e, 1 | H | e, 1 \rangle & \langle e, 1 | H | e, 0 \rangle & \langle e, 1 | H | g, 0 \rangle \\ \langle e, 0 | H | e, 1 \rangle & \langle e, 0 | H | e, 0 \rangle & \langle e, 0 | H | g, 0 \rangle \\ \langle g, 0 | H | e, 1 \rangle & \langle g, 0 | H | e, 0 \rangle & \langle g, 0 | H | g, 0 \rangle \end{pmatrix}$$

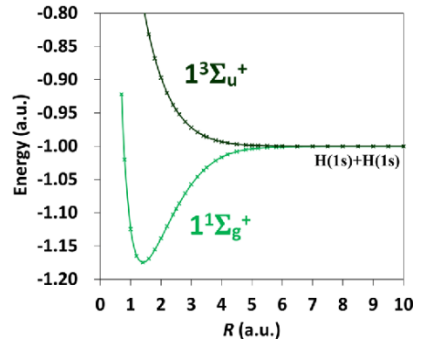
C. Now let's include spontaneous emission, which permits decays from $|e, n\rangle$ to $|g, n\rangle$ by emitting a photon into free space. Draw wiggling lines to show all possible spontaneous emission toward lower eigenstates. Determine the frequencies of all emitted photons if an atom is initially prepared in any of the lowest 5 eigenstates.

2. Hyperfine structure of a H₂ molecule

Hamiltonian of a H₂ molecule in the electronic ground state $n = 1$ can be written as

$$H = A(s_1 \cdot i_1 + s_2 \cdot i_2) + V(R),$$

where $s_1 = \frac{1}{2}$, $s_2 = \frac{1}{2}$, $i_1 = \frac{1}{2}$, $i_2 = \frac{1}{2}$ are the electron spins and nuclear spins of the 1st and 2nd atom and R is the interatomic separation. The molecular potential is conventionally modeled by the sum of a spin-independent scalar potential $V_s(R) \approx -\frac{C_6}{R^6} + \frac{C_{12}}{R^{12}}$ and a spin-dependent exchange term $V_{ex}(R)s_1 \cdot s_2$ that emerges at a much shorter range.



$$V(R) = V_s(R) + V_{ex}(R)s_1 \cdot s_2$$

The figure shows the experimentally measured molecular potential in atomic units (a.u.), which includes the singlet potential $V_1(R)$ labelled as $1^1\Sigma_g^+$ with $S = s_1 + s_2 = 0$ and triplet state $V_0(R)$ labelled as $1^3\Sigma_u^+$ with $S = 1$.

A. Assuming the hyperfine interaction is negligible $A \approx 0$, determine V_s and V_{ex} based on V_0 and V_1 .

B. Determine the hyperfine structure of H₂ molecules in the singlet and triplet ground states. (Hint: Here the hyperfine interactions is weak compared to the exchange energy $A \approx 10^{-6}$ in atomic units. You may evaluate the hyperfine energy perturbatively.)

C. Determine the hyperfine structure of highly excited H₂ molecules with sizes > 10 a.u. (Hint: At large interatomic separations, as you can see in the figure, the atomic interaction is negligible $V_{ex}, V_s \approx 0$ and the Hamiltonian is dominated by the hyperfine interactions.)

3. Low energy scattering with a square well

The simplest model to describe finite range interactions between atoms is a 3D square well

potential $V(r < r_0) = -\frac{\hbar^2 q^2}{2\mu} \equiv -D$ and $V(r > r_0) = 0$. Consider the incoming spherical wave

e^{-ikr} / r and the outgoing wave is $S e^{ikr} / r$, where $S = e^{2i\delta}$ is the scattering matrix and δ is the s-wave scattering phase shift.

A. Starting with Schrodinger's equation in the spherical coordinate, and show that the scattering

phase shift is given by $\delta = -kr_0 + \tan^{-1} \frac{k \tan \sqrt{q^2 + k^2} r_0}{\sqrt{q^2 + k^2}}$.

B. Determine scattering length a in the low scattering energy limit $k \rightarrow 0$.

(Hint: Determine the location where wavefunction vanishes in the low energy limit.)

C. Plot the scattering phase shift and scattering length vs. the depth D . Determine the depth $D = D_n$ when the n -th molecular state emerges below the continuum.