Physics 471 – Introduction to Modern Atomic Physics

HOMEWORK 5 (Due: 2/19/2024)

1. Two-body entanglement

Consider two qubits A and B, the wavefunction can be described by 4 probability amplitudes as $|\Psi>= a_{00}|00> +a_{01}|01> +a_{10}|10> +a_{11}|11> \equiv \begin{pmatrix} a_{00} \\ a_{01} \\ a_{10} \\ a_{11} \end{pmatrix}$, and a_{ij} are complex probability amplitudes.

A product state of the qubits is defined as the direct tensor product of the individual qubit states, and is thus separable and not entangled.

$$|\text{Product state}>= |\psi_A> \otimes |\psi_B>$$

The Bell states are considered as the maximally entangled states of two qubits, defined as

$$|\Phi^{\pm}>= 2^{-\frac{1}{2}}(|00> \pm|11>) = 2^{-1/2}\begin{pmatrix} 1 \\ 0 \\ 0 \\ \pm1 \end{pmatrix}$$

$$|\Psi^{\pm}>= 2^{-\frac{1}{2}}(|01> \pm|10>) = 2^{-1/2}\begin{pmatrix} 0 \\ 1 \\ \pm1 \\ 0 \end{pmatrix}$$

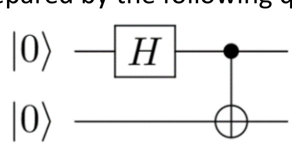
The Bell states form a complete basis to describe all possible pure states of 2 qubits.

A. Express the following states in the Bell basis:

$$|a>= 2^{-1}(|0> +|1>)\otimes(|0> -|1>)$$

$$|b>= 2^{-\frac{1}{2}}(|00> +i|11>)$$

B. Show that $|\Phi^{+}>$ can be prepared by the following quantum circuit

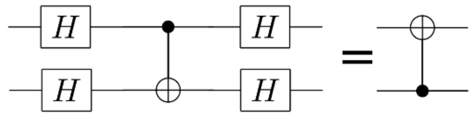


where the Hadamard gate *H* is followed by the controlled-NOT gate:

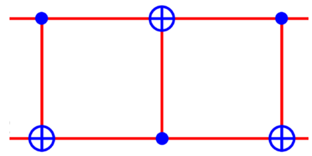
$$\text{CNOT}=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

2. Prove the following equivalence of logic gate operations

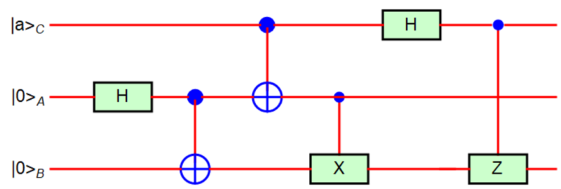
A.



B. SWAP gate exchanges the states of two qubits as $S|\psi_A > \otimes |\psi_B >= |\psi_B > \otimes |\psi_A >$.
Show that $S = \frac{1}{2}(1 + \vec{\sigma}_A \cdot \vec{\sigma}_B)$, where $\vec{\sigma}_A$ is the Pauli matrix acting on the qubit A and is equivalent to the following circuit.



C. Quantum teleportation



Charle holds a qubit in quantum state $|a >= \binom{\alpha}{\beta}$ and wishes to transfer the state to Bob. To teleport the state, Alice and Bob begin with $|0>$ and $|0>$ and then share an EPR pair of qubits with the first 2 operations. Show the subsequent operation allows the transfer of the state $|a>$ to Bob.

3. First and second quantization

Assume we have 3 bosons, and two are in the ground state $|0>$ and one in the excited state $|1>$. The fully symmetrized wavefunction in the 1st quantization form is given by

$$|\psi_i>= 3^{-\frac{1}{2}}(|0,0,1>+|0,1,0>+|1,0,0>),$$

where $|0,0,1>\equiv|\psi_1=0>\otimes|\psi_2=0>\otimes|\psi_3=0>$ is the direct tensor product of the quantum states of 3 particles.

Let's apply a pulse to the bosons with the Hamiltonian $H=\frac{\hbar\Omega}{2}(\sigma_x\otimes 1\otimes 1+1\otimes\sigma_x\otimes 1+1\otimes 1\otimes\sigma_x)$ for a duration of $t_\pi=\frac{\pi}{\Omega}$ (π- pulse) or $t_{\pi/2}=\frac{\pi}{2\Omega}$ ($\pi/2$-pulse).

A. What is the final state of the bosons $|\psi_f>$ after the pulses?

Now let's reformulate the above in the 2nd quantization form. The initial state can be written as $|\Psi_\mathrm{i}>=|n_g=2,n_e=1>$, where there are n_g particles in the ground state and n_e particles in the excited state.

B. Write the final state the pulses in the 2nd quantization form $|\Psi_\mathrm{f}>$ in the basis of $|n_g,n_e>$.

C. The Hamiltonian in the 2nd quantization form is $H=\frac{\hbar\Omega}{2}(a_g^+a_e+a_e^+a_g)$, evaluate $|\Psi(\mathrm{t})>$ and show that you derive the same final state after pulses as in problem A.

(Hint: Note that the Hamiltonian conserves the total particle number $N=a_g^+a_g+a_e^+a_e$ and thus the wavefunction evolves in the restricted Fock space as $|\Psi>=A_{30}|3,0>+A_{21}|2,1>+A_{12}|1,2>+A_{03}|0,3>$. You can solve $|\Psi_\mathrm{f}>=e^{-\frac{iHt}{\hbar}}|\Psi_\mathrm{i}>$ by diagonalizing the Hamiltonian.)