Physics 471 – Introduction to Modern Atomic Physics

HOMEWORK 6 (Due: 2/26/2024)

1. Stability of bosons in the ground state

Before the experimental observation, many considered Bose-Einstein condensation will attract all bosons toward a single quantum state and effectively form a blackhole. Here we consider the stability of the BEC based on N bosons with mass m in a large box with volume $V = L^3$.

A. Using the Gross-Pitaevskii equation, $\left(-\frac{\hbar^2\nabla^2}{2m} + V(x) + g|\psi|^2\right)\psi = \mu\psi$, where $V(x) = 0$ inside the box and infinity otherwise, show that for $g = \frac{4\pi a\hbar^2}{m} > 0$ the chemical potential approaches $\mu_1 = \frac{3h^2}{8mL^2}$ in the low density limit and $\mu_2 = \frac{4\pi a\hbar^2 N}{mL^3}$ in the high density limit. Argue that in the general case the chemical potential is $\mu \approx \mu_1 + \mu_2 > \mu_1, \mu_2$.

B. For bosons with negative scattering length $g = \frac{4\pi a\hbar^2}{m} < 0$, argue that the system becomes unstable (and collapses) when chemical potential drops below zero $\mu \leq 0$. Estimate the critical scattering length and sketch the density distribution $|\psi|^2$ when the BEC collapses (For simplicity, you can consider BEC in 1D.)

2. Spontaneous symmetry breaking

Here we consider a simple model of spontaneous symmetry breaking. Consider N bosons with the energy in the 2nd quantization form

$$H = \sum_k \epsilon_k a_k^+ a_k + \frac{g}{2} \sum_{k_1,k_2,k_3} a_{k_3}^+ a_{k_1+k_2-k_3}^+ a_{k_1} a_{k_2} \,,$$

where ϵ_k is the kinetic energy of a boson with momentum k, a_k and a_k^+ are the annihilation and creation operators of the boson with commutator $\left[a_k, a_{k'}^+\right] = \delta_{kk'}$, and the summations go over all momenta $-\infty < k_i < \infty$. $g > 0$ is the coupling constant.

A. Assume the kinetic energy has a single minimum $\epsilon_k = k^2$ at $k = 0$. Show that the ground state is $|g >= |n_{k=0} = N, n_{k\neq 0} = 0 >$ and the ground state energy is $< H > = < g|H|g > = \frac{g}{2} N(N-1)$, where N is the particle number.

B. Assume the kinetic energy has 2 mimina $\epsilon_k = (k^2 - 1)^2$ at $k = \pm 1$, do bosons condense into one of the minima or a superposition of the two?

Let's start with the assumption N_+ atoms have momentum $k = 1$ and N_- atoms have momentum $k = -1$, and the total number is conserved $N_+ + N_- = N$. Thus the system is described by $|N_+, N_- >$. Evaluate the energy of the system $< H >=< N_+, N_-|H|N_+, N_- >$ and determine the combination of N_+, N_- that gives the lowest energy?

C. Have we found the ground state in B? Assume $N = 3$, and thus the ground state can be expressed in the basis of $|3, 0 >$, $|2, 1 >$, $|1, 2 >$ and $|0,3 >$. Minimize the energy and compare the ground state with the result you got in B.?

D. Can you draw a conclusion from the above on the condition that bosons would spontaneously break the reflection symmetry $k \leftrightarrow -k$ at low temperatures when there is degeneracy in the ground state?