DFT and FT

When discrete, equally spaced data is given as $x_0, x_1 \cdots x_{N-1}$

fast Fourier transform is defined as

$$K_j = \sum_{l=0}^{N-1} x_l \, e^{-i2\pi j l/N} \qquad j = 0 \cdots N-1$$

inverse FFT is

$$x_l = \frac{1}{N}\sum_{j=0}^{N-1} K_j \, e^{i2\pi j l/N} \qquad l = 0 \cdots N-1$$

This equation defines the meaning of K_j :

$\frac{K_j}{N}$ is the coefficient of the term $e^{i2\pi j l/N} = e^{i2\pi \frac{l}{N/j}}$ whose freq is $2\pi j/N$

If the step size of x_l is Δx, the period is $\frac{N}{j}\Delta x$, freq $2\pi j/N\Delta x$

Application : Given a discrete sample of $f_l = f(\Delta t\, l)$, $l = 0 \cdots N-1$

What do we know about its spectrum $f(\omega) = \frac{1}{2\pi}\int f(t) e^{-i\omega t} dt$

from the discrete FT $A_j = \sum_{l=0}^{N-1} f_l \, e^{-i2\pi j l/N}$?

$$f(t) = \int f(\omega) e^{i\omega t} d\omega$$

$$f_l = \int f(\omega)\, e^{i\omega l \Delta t} d\omega$$

$$A_j = \sum_{l=0}^{N-1} \int f(\omega) e^{i\omega l\Delta t}\, e^{-i2\pi j l/N} d\omega$$

$$= \int f(\omega) \sum_{l=0}^{N-1} e^{i\omega \Delta t l} e^{-i2\pi j l/N} d\omega$$

$$= \int f(\omega) \frac{1-e^{i\alpha N}}{1-e^{i\alpha}} d\omega$$

$$\equiv \int f(\omega)\, g(\omega_j - \omega)\, d\omega$$

$$= \int f(\omega)\, e^{i(N-1)\alpha/2} \frac{\sin N\alpha/2}{\sin \alpha/2} d\omega$$

$$= \int f(\omega)\, g(\omega - \omega_j)\, d\omega$$

$$\text{FWHM} = \frac{2\pi}{N} \Rightarrow \Delta\omega = \frac{2\pi}{N\Delta t}$$

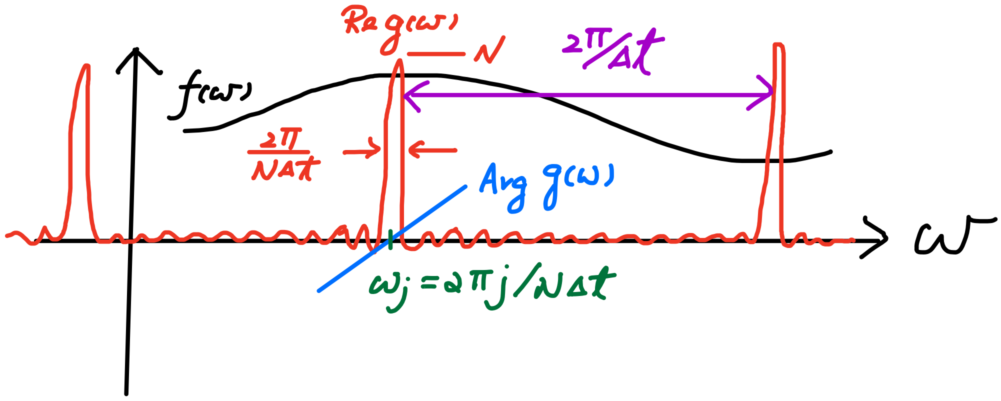
$$\alpha \equiv \omega\Delta t - 2\pi j/N$$

$$\sum_{l=0}^{N-1} e^{i\alpha l} = \frac{1-e^{i\alpha N}}{1-e^{i\alpha}}$$

$$\omega_j = 2\pi j/N\Delta t$$

$$g = \frac{e^{iN\alpha/2}(e^{iN\alpha/2} - e^{-iN\alpha/2})}{e^{i\alpha/2}(e^{i\alpha/2} - e^{-i\alpha/2})}$$

$$= e^{i(N-1)\alpha/2} \frac{\sin N\alpha/2}{\sin \alpha/2}$$



Assume $f(\omega)$ is a slow-varying function

$$\Rightarrow A_j = \frac{2\pi}{\Delta t} \sum_{K=0,\pm 1,\dots} f(\omega_j + 2\pi K/\Delta t) \qquad \omega_j = 2\pi j/N\Delta t$$

For $0 \le \omega_j \le \pi/\Delta t$ $\quad A_j \approx \frac{2\pi}{\Delta t}[f(\omega_j) + f(\omega_j - 2\pi/\Delta t)]$

$$= \frac{2\pi}{\Delta t}[f(\omega_j) + f^*(\omega_{N-j})]$$

(For real f_ℓ) $\Rightarrow A_{N-j} = \frac{2\pi}{\Delta t}[f(\omega_{N-j}) + f^*(\omega_j)]$

$$A_j = A_{N-j}^*$$

Conclusion

Given a discrete data set of length N: $f_\ell = f(x = \ell\Delta x)$, $\ell = 0 \dots N-1$

DFT or FFT gives $A_j = \sum_{\ell=0}^{N-1} f_\ell \, e^{-i2\pi j\ell/N}$

$$= \int f(\omega) e^{i(N-1)\alpha/2} \frac{\sin N\alpha/2}{\sin \alpha/2} d\omega \qquad \alpha = (\omega - \omega_j)\Delta t, \quad \omega_j = 2\pi j/N\Delta t$$

$$\approx \frac{2\pi}{\Delta t}[f(\omega_j) + f^*(\omega_{N-j})]$$