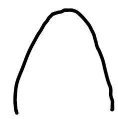
Scale invariance and universality

 $\vec{x}(t) = (v_x t,\ v_y t - \frac{1}{2} g t^2)$

System remains invariant when we do the following transformation:



$x \to \lambda x$
$t \to \lambda x$
$g \to g/\lambda$

Simulate a giant robot walking on earth

$\Rightarrow$ $m \partial_t^2 x = -mg$

$x \to \lambda x$
$t \to \lambda^{1/2} t$
$v \to \lambda^{1/2} v$

Simulate hypothetical planet motion

$\Rightarrow m \partial_t^2 r = -\frac{GMm}{r^2}$

$r \to \lambda r$
$M \to M$
$t \to \lambda^{3/2} t$

or

$r \to \lambda r$
$M \to \lambda^3 M$
$t \to t^0$

Kepler's law

Motion in viscous medium

$\Rightarrow m \partial_t^2 x = -mg - \epsilon v A$

$x \to \lambda x$
$t \to \lambda^{1/2} t$
$m \to \lambda^3 m$
$v \to \lambda^{1/2} v$

not scale invariant unless

$\epsilon \to \lambda^{1/2} \epsilon$

$\Rightarrow$

Liquid friction is 1000 times larger than air $\Rightarrow$

Human swimming $\approx$ bacteria floating in air

HW1: identify the eqn that describes water waves. How would you faithfully simulate ~100m scale ocean waves in a ~1m water tank?

Interacting particles

$$i\hbar\partial_t\varphi = H\varphi = \left[\Sigma_i \frac{p_i^2}{2m} + V\right]\varphi$$

$$x\to\lambda x,\ p\to\lambda^{-1}p,\ t\to\lambda^2 t,\ \underline{\underline{V = \lambda^{-2}V}}$$

1st option: $V(x)\to V(\lambda x) = \lambda^{-2}V$

$$\Rightarrow V(x) = \frac{A}{x^2}.$$ Example: Efimov potential

2nd option: $V(p)\to V(\lambda^{-1}p) = \lambda^{-2}V \Rightarrow V = p^2$ what's this?

Also $V = \frac{p}{x},\ p^3x,\ \frac{1}{x^2}e^{ipx}$

3rd: Short range interaction $V = g|\varphi|^2$

$$g'|\varphi'|^2 = g'|\varphi|^2\lambda^{-D} = \lambda^{-2}g|\varphi|^2$$

$$\Rightarrow g' = g\,\lambda^{D-2}$$

$D=2$: $g = \text{const}$: $\underline{\text{2D gas is scale invariant}}$.

$D=3$: $g' = g\lambda$: $g = x,\ \underline{\frac{1}{p}},\ n^{-1/3}\ldots$

$D=1$: $g' = g\lambda^{-1}$: $g = \frac{1}{x},\ p,\ \underline{\rho\ (V = \rho^2)}$

Two famous examples:
2D gas with constant interaction (Stringari, 1997)
Unitary Fermi gas (T.L. Ho, 2000)

HW 2: Identify two interesting scale invariant systems with interacting particles with linear energy dispersion E= p

Example: 2D gas.

$$i\hbar\partial_t\varphi = \left[\frac{\hbar^2}{2m}\nabla^2 + g|\varphi|^2\right]\varphi\,. \quad g = \text{const}.$$

given $\varphi(x,0)$ determine $\varphi(x,t)$

$$\begin{aligned} x &\to \lambda x \\ t &\to \lambda^2 t \end{aligned} \Rightarrow |\varphi| \to \lambda^{-1}|\varphi|$$

$$|\varphi(\lambda x, \lambda^2 t)| = \lambda^{-1}|\varphi(x,t)|. \quad \text{Choose } \lambda = t^{-1/2}$$

$$|\varphi(x,t)| = \frac{1}{\sqrt{t}} f\left(\frac{x}{\sqrt{t}}\right) \Rightarrow \text{reduce to ODE}.$$

This is similar to heat transport equation.

Consequence of scale invariance

$$\begin{aligned} x &\to \lambda x \\ p &\to \lambda^{-1} p \\ E \sim p^z &\to \lambda^{-z} E \\ t \sim \frac{1}{E} &\to \lambda^{z} E \end{aligned} \qquad \begin{aligned} &\Rightarrow n \to \lambda^{-D} n,\ \varphi \to \lambda^{-D/z}\varphi \\ &\Rightarrow \mu \to \lambda^{-z}\mu,\ T \to \lambda^{-z} T \end{aligned}$$

$$n = n(\mu, T) \Rightarrow \lambda^{-D} n = n(\lambda^{z}\mu, \lambda^{-z} T)$$

$$\Rightarrow n(\lambda^{-z}\mu, \lambda^{-z} T) = \lambda^{-D} n(\mu, T)$$

if we choose $\lambda^{z} = kT \Rightarrow n(\mu, T) = \lambda^{D} n(\mu/kT, 1)$

$$\Rightarrow n(\mu, T) = T^{D/z} F(\mu/T)$$

We can do this on any function

$$dE = TdS + \mu dN - pdV = 0 \Rightarrow p = Tds + \mu dn$$

$$\begin{aligned} \Rightarrow P &= \int n\, d\mu \quad (T = \text{const.}) \\ &= T^{D/z+1} \int^{\mu/T} F(x)\, dx \\ &= T^{D/z+1} \mathbb{F}\left(\frac{\mu}{T}\right), \quad \mathbb{F}'(x) = F(x) \end{aligned}$$

HW3 show that entropy density s and total energy density e are given by

$$\frac{S}{N} = s = T^{D/z}\left[\left(\frac{D}{z}+1\right)\mathbb{F}\left(\frac{\mu}{T}\right) - \frac{\mu}{T}F\left(\frac{\mu}{T}\right)\right]$$

$$\frac{E}{N} = \varepsilon = T^{D/z+1}(D/z)\,\mathbb{F}\left(\frac{\mu}{T}\right) = n^{1+z/D}\, G\left(\frac{s}{n}\right).$$ determine G.