What determine the coupling constant $g$?

Scattering theory

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incident wave
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\[ \text{wave function} = e^{ikz} + f(q) \frac{e^{ikr}}{r} = e^{ikz} + \sum_{lm} \lambda_{lm} Y_{lm}(\theta, \phi) \frac{e^{ikr}}{r} \]

Low temperature limit: only $f_{00}$ is non-zero (s-wave)

Only s-wave (head-on collisions) is left, we have

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incident wave
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We get $y(r) = \frac{e^{ikr}}{r} - \sum \frac{e^{ikr}}{r}$

Scattering matrix $S = e^{i2\delta}$

\[ = \frac{1}{r} e^{i\delta} \sin(kr + \delta) \]

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scattering energy $E_s \to 0$
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with fugal barrier $\frac{L(L+1)}{2mr^2} >> E_s$
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scattering phase shift
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in the spherical coordinate
Interaction between 2 atoms in the ground state:

Energy

\[ H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + V(x_1) + V(x_2) + U(1x, x_2) \]
\[ = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + V(x_1) + V(r) + U(r) \]

Since trap size is typically \( \gg \) molecular potential length,
\[ H_r = -\frac{\hbar^2}{2m} \nabla^2 + V(r) + V_C(r) \]

Consider 2 atoms in a trap.

\[ \psi(r) \neq 0 \quad \text{for} \quad r \gg a \]

\[ \psi \to 0 \quad \text{for} \quad r \ll a \]

\[ \psi \sim \exp(-ikr) \quad \text{for} \quad r \to \infty \]

\[ \psi \sim \exp(ikr) \quad \text{for} \quad r \to 0 \]

\[ \psi \sim c \text{ const.} \quad \text{for} \quad r \approx a \]

Note that we have assumed s-wave scattering implicitly!
S-wave scattering length: offset of the $l=0$ radial wavefunction.

\[
\lim_{r \to 0} \lim_{k \to 0} \frac{rY^0}{(rY)^0} = \lim_{k \to 0} \frac{\sin kr + \delta}{k + \cos kr + \delta} = \lim_{k \to 0} \frac{\tan \delta}{k} = -\frac{a}{\hbar^2}
\]

\[
\lim_{k \to 0} \frac{\psi}{r} \rightarrow \frac{A}{r} (r-a) = A (1- \frac{a}{r})
\]

\[-\frac{h^2}{2m} \nabla^2 \psi = -\frac{h^2}{2m} (-Aa) \nabla^2 \frac{1}{r} = -\frac{h^2}{m} a 4\pi \delta(r) A
\]

\[-\frac{h^2}{2m} \nabla^2 \psi + \frac{4\pi a \hbar^2}{m} \delta(r) \psi = 0.
\]

\[\Rightarrow \text{This is the Schrödinger's eqn with an effective interaction potential } g \delta(r) !
\]

Final result: Coupling constant $g = \frac{4\pi a \hbar^2}{m}$

\[
\mathcal{H} \psi(x) = \left[ \frac{p^2}{2m} + V(x) + \frac{4\pi a \hbar^2}{m} 12\psi \right] \psi
\]

Additional assumptions we have made:

1. Short-range interaction
2. Low-temperature scattering (S-wave scattering)

References: Modern Quantum mechanics, J.J. Sakurai, Chapter 7
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