Physics 452 - Quantum Optics and Quantum Gasses

Final

(Due: 10am, 12/12/2017 @ CIS E107)

1. Quantum Apples in Eden

In the grand opening of the Garden of Eden, new types of apples are produced in two species: |edible> or $2^{-1/2}$(|edible>+|poisonous>), which cannot be distinguished before eating. Eve bought a couple apples as gifts to Snow White.

A. Snow White grabs an apple and eats it, what is the chance she is poisoned?

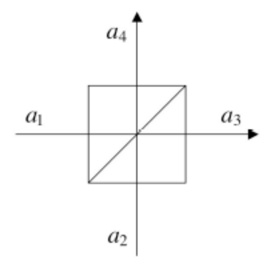
B. Snow White survived in the first round. Eve decides to give her whole bunch of apples of the same type. As Snow White's royal wizard or wizardess, you can either perform some operations on the apples, or feed them to her 7 dwarf friends. How would you do to make absolutely sure at least one is edible for Snow White? Or can you?

(Hint: A wizard can perform arbitrary unitary operations on the apples, and you have antidotes for the poison, so no dwarf dies.)

C. Prove that you cannot distinguish two states with certainty if they are not orthogonal.

D. You are fired, and now work for Eve. Your new task is to make a 100% |poisonous> apple by entangling two $2^{-1/2}$(|edible>+|poisonous>) apple. How would you do that?

2**. Beam splitter**



Here we consider a 4-port beam splitter. In typical operation, an incident mode labeled 1 gives rise to a reflected mode 4 and transmitted mode 3. The amplitude transmissivity and reflectivity are in general complex numbers given by *t* and *r*. An incident beam from port 2 has transmissivity *t'* and reflectivity *r'*.

A. Show that if there are no losses in the beam splitter, the parameters satisfies the reciprocity relations:

$|t|=|t'|$, $|r|=|r'|$, $|t|^2+|r|^2=1$, and $rt^*+r'^*t'=0$.

Construct a matrix M that associates the output field $E_{out}=(E_3,E_4)^T$ to the input field $E_{in}=(E_1,E_2)^T$ given by $E_{out}=ME_{in}$. Show that the matrix is unitary and write down a simple example of *M* for the special case of $|r|^2=|t|^2=50\%$.

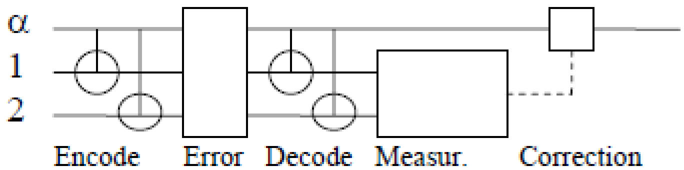
B. Now go to the quantum regime and replace E_i by operator $\hat{a}_i$. Given that $[\hat{a}_1,\hat{a}_1^+]=[\hat{a}_2,\hat{a}_2^+]=1$, show that $[\hat{a}_3,\hat{a}_3^+]=[\hat{a}_4,\hat{a}_4^+]=1$. Show that even in the case when you have no light (vacuum field) at one of the input ports, the vacuum fluctuations are essential for the consistency of the bosonic statistics of the output fields.

C. Rewrite the equivalence of $E_{out}=ME_{in}$ in the quantum regime and show that $[\hat{a}_3,\hat{a}_4^+]=0$ and derive the observables $\hat{n}_3$ and $\hat{n}_4$.

D. Finally we consider the input state is a dual single photon state at each of the incident ports $|in>=|1,1>_{12}$. Show that $|\,out>=(|t|^2-|r|^2)\,|1,1>_{34}+i\sqrt{2}\,|rt|\,(|2,0>_{34}+|0,2>_{34})$.

Comment on the outgoing state for a balanced beam splitter with $|r|^2=|t|^2=50\%$.

3. Simple quantum error correction



A qubit is in the state of $|\alpha\rangle = a\,|0\rangle + b|1\rangle$. ($|a|^2+|b|^2=1$) Two ancilla qubits 1 and 2 are initially in state $|0\rangle$. They will help to witness and correct for potential errors occurring on the qubit.

In the encoding phase, two control-not gates (CNOT) operate on the ancilla qubits. Here a CNOT gate flips the target spin (1 or 2) when the control spin (α) is 1. When the control spin is 0, the target spin is unchanged.

After Encoding, the system may pick up error from the environment in the Error phase. After two more CNOT gates in the Decoding phase, we perform measurement on the ancilla qubits. Finally we correct the error on the qubit based on the measurement result.

A. Show that the encoding operation creates a GHZ state $a|000\rangle + b|111\rangle$.
B. Assume an error is introduced to the system as an addition of $\varepsilon|\ \rangle$' to the whole system. In the absence of error, $\varepsilon=0$, show that the qubit comes back to $a\,|0\rangle + b|1\rangle$.
C. What would the state of the system after the decoding phase if the noise amplitude is $|\ \rangle' = |100\rangle$, $|010\rangle$ or $|001\rangle$ for example?
(Hint: You may assume $\varepsilon \ll 1$ and provide proper normalization if necessary.)
D. Based on these examples, if you measure the state of the ancilla qubits, how would you know the system is erred? Without any correction, what is the probability that the system remains in the initial state?
E. Given the measurement result (classical information), how would you use it in the Correction phase to improve the fidelity of the system (reduce the error rate)?
F. What is the success rate of your strategy if the noise is a random vector in the Hilbert space of $|\alpha, 1, 2\rangle$?

4. Entanglement and no communication theorem

Let's start with a pure singlet state $|s\rangle = 2^{-1/2}(|1,0\rangle - |0,1\rangle)$, where the 2 indexes refer to spin *a* and *b*, respectively.

A. Determine the density matrix $\rho_s = |s\rangle\langle s|$ and prove that it cannot be written in the form of $\rho_s = \sum_i^N p_i \rho_a^i \otimes \rho_b^i$, where $\rho_a^i = |i\rangle_a\langle i|_a$ is the density matrix of spin a in the i-th arbitrary state $|i\rangle_a$, $\rho_b^i = |i\rangle_b\langle i|_b$ is the density matrix of spin b in the i-th arbitrary state $|i\rangle_b$, $0 \le p_i \le 1$ is the classical probability, and $\sum_i p_i = 1$.

(Hint: in the basis of |0,0>, |0,1>, |1,0> and |1,1>, $\rho_s = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.)

B. Now Alice (a) and Bob (b) share many pairs of spins in the singlet state. Bob wishes to construct the density matrix of his particle. Show that no matter how he rotates and measures his spin, he concludes $\rho_b = tr_a[\rho_{ab}] = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$, which looks like a classical mixture.

C. Show that the above result remains unchanged, if Alice has measured her spin prior to Bob's measurement.
(Hint: The result is the same, but the math is different.)

D. Show that the entropy of the system is zero $S_{ab} = -Tr[\rho_{ab} \log \rho_{ab}]$=0, while entropy of the sub-system $S_a + S_b = -Tr[\rho_a \log \rho_a] - Tr[\rho_b \log \rho_b]$ is > 0,
Many consider this the reason that entropy is increasing with time in an open system. *In the grand scheme of things*, entropy of the world is a constant.

5. Fano profile

Ugo Fano (1912-2001) was Fermi's student and a UChicago professor on atomic physics for 35 years. Here we discuss the Fano profile that he described in Physical Review 124, 1866 (1961), which is observed in all sub-fields of physics and chemistry.

In the simplest form, the Fano profile shows that the cross section of a scattering event with a tunable energy E can be interfered by a nearby quantum state with energy E_{res} such that

$\sigma(\varepsilon) = \sigma_{bg} \frac{(q+\varepsilon)^2}{1+\varepsilon^2}$, where $\varepsilon = \frac{E - E_{res}}{\gamma/2}$, γ is the width, q is called the Fano parameter.

The characteristic features include a resonance at $\varepsilon = -q$ and an interference pole at $\varepsilon = 0$

Let's review a few cases that show Fano or Fano-like profile:

Case 1: Collisions between cold atoms near a Feshbach resonance (Lecture 7)

Elastic cross section between atoms is $\sigma = \frac{4\pi}{k^2} \sin^2 \delta$

Collision phase shift $k \cot \delta = -\frac{1}{a}$

Scattering length $a = a_{bg}(1 - \frac{\Delta}{B - B_{res}})$. Show that the atomic collision cross section $\sigma(B)$ displays a Fano profile near the Feshbach resonance. Identify the location of the peak, pole and the Fano parameter?

Case 2: Electromagnetically induced transparency (Homework 6)

In a 3-level system coupled by a control field and a probe field, you may consider a finite and fixed probe beam detuning Δ_p, tune the control beam detuning Δ_C, and monitor the absorption or the energy dissipation of the probe beam. Determine the peak, zero and the Fano parameter.

Case 3: Dissociation of molecules
A Feshbach molecule formed at a large scattering length a >0 has a simple wavefunction of $r\psi_m(r) = \sqrt{\frac{2}{a}} e^{-\frac{r}{a}}$. By suddenly quenching the scattering length to a new scattering length a', the molecules can be dissociated into free atoms in the continuum, described by scattering state: $r\psi(r) = A\sin(kr + \delta')$, where $k \cot \delta' = -1/a'$. The probability of dissociation is given by the so called Franck-Condon factor $F_C = \int \psi^*(r)\psi_m(r) d^3 r$. Show that F_C develops a Fano profile as a function of a'. Determine the locations of peak, pole and Fano parameter.

Case 4: Come up and describe another simple model that shows Fano profile in quantum physics, E&M, HEP, chemistry…. or even in classical physics. Prove it.