Physics 452 - Quantum Optics and Quantum Gasses Cheng Chin

HOMEWORK I (Due: 10/04/2017)

1. (math) Variation calculation

Lagrangian density of the Klein-Gordon field is given by

$L(\psi,\nabla\psi)=\frac{1}{2}\dot{\psi}^2-\frac{1}{2}\nabla\psi\cdot\nabla\psi-\frac{1}{2}m^2\psi^2$,

Given action $S=\int dxdtL$, show that $\delta S=0$ yields the equation of motion,

$\partial_t^2\psi-\partial_x^2\psi+m^2\psi=0$, of a relativistic spinless particle.

2. Laser cooling and the cooling limit?

An atom is cooled by laser when it scatters photons and loses its kinetic energy. In a scattering process, atom absorbs a photon from the incident laser beam and spontaneously emits a photon in a random direction. After such an event, the energy of the atoms becomes

$E=\frac{(\vec{p}+\hbar\vec{k}+\hbar\vec{k}_s)^2}{2m}$, where $\vec{p}$ is the initial momentum of the atom, m is the atomic mass, $h\vec{k}$ is the momentum of the incident photon, and $h\vec{k}_s$ is the momentum of the emitted photon. For simplicity, we assume $|\vec{k}_s|=|\vec{k}|\equiv k$.

A. If you have a way to always precisely point $\vec{k}$ in the opposite direction of the atomic momentum, and the initial atom energy is very high $E_0=\frac{3}{2}k_BT_0>>E_R$, where $E_R\equiv\frac{(\hbar k)^2}{2m}$ is the recoil energy, calculate the expectation value of the cooling trajectory *<T(N)>*, where *N* is the number of photons scattered by the atom.

B. What is the expected temperature after scattering many photons?

3. Dilute Bose-Einstein condensate in a trap

Consider a condensate of *N* bosons at *T*=0 described by the Gross-Pitaevskii equation,

$\mu\Psi(r)=(\frac{p^2}{2m}+V(r)+g|\Psi(r)|^2)\Psi(r)$, where $V(r)=\frac{1}{2}m\omega^2r^2$ is an harmonic potential,

$g=\frac{4\pi\hbar^2}{m}a$, and a is the scattering length.

a. use Thomas-Fermi approximation, and show that the chemical potential μ is given by

$\mu=\frac{15^{2/5}}{3}(\frac{Na}{\bar{a}})^{2/5}\hbar\omega$, where $\bar{a}=\sqrt{\frac{\hbar}{m\omega}}$ is the harmonic oscillator length.

b. show that interaction energy $U \equiv < gn >= \frac{3}{7}\mu$, potential energy $< V >= \frac{4}{7}\mu$ and the total energy of the system is $E = \frac{5}{7}\mu N$. Thus we have $< gn >:< V >: \frac{E}{N} : \mu$ $< gn >:< V >: \varepsilon : \mu$ =3:4:5:7.

(Hint: chemical potential is $\mu = \frac{\partial E}{\partial N}$.)

4. Bose condensate near a wall

A. Given a wall $V(r < 0) = \infty$ and $V(r > 0) = 0$, calculate the density of a condensate with chemical potential μ by solving Gross-Pitaevskii equation,

$$\mu\Psi(r) = (\frac{p^2}{2m} + V(r) + g\,|\,\Psi(r)\,|^2)\Psi(r)\,.$$

B. We know that single particle ground state in a box is $\Psi(x) \sim \sin kx$. How would you modify the potential such that the ground state wavefunction of a single particle is $\Psi(x) \approx$const.?