1. Single channel scattering in square well

Given a square well potential in the spherical coordinate \( V(r < r_0) = -\frac{\hbar^2 q^2}{2\mu} \equiv -D \) and \( V(r > r_0) = 0 \), the incoming spherical wave is \( e^{-ikr} / r \) and the outgoing wave is \( Se^{ikr} / r \), where \( S = e^{2i\delta} \) is the scattering matrix and \( \delta \) is the s-wave scattering phase shift.

A. show that the scattering phase shift is 
\[
\delta = -kr_0 + \tan^{-1} \left( \frac{k \tan \sqrt{q^2 + k^2 r_0}}{\sqrt{q^2 + k^2}} \right).
\]

B. Determine scattering length \( a \) in the low scattering energy limit \( k \to 0 \).

C. Plot the scattering phase shift and scattering length vs. the depth \( D \).

D. What is the value of scattering phase shift when the scattering length diverges?

2. Bound states and potential resonance

Continue 1 with the square potential. Here we will look into the connection between the scattering length and the bound state.

A. When we increase \( D \) from zero, show that the condition for the potential to support one more bound state is the same as the condition you derived in 1 D.

B. In particular, when the scattering length is very large and positive \( a >> r_0 \), show that there is a weakly bound state near the dissociation threshold with energy
\[
E = -\frac{\hbar^2}{2\mu(a - r_0)^2}.
\]

C. According to molecular spectroscopy, there are 153 bound states of a diatomic cesium molecule Cs\(_2\) in the singlet potential. What is the singlet scattering phase shift of two colliding cesium atoms in the low temperature limit?