1. Eigenvalue-eigenfunction vs. 2nd quantized form
Consider a Hamiltonian given by
\[ \hat{H} = \alpha (\hat{a}^+ \hat{a} + \hat{b}^+ \hat{b}) + \beta (\hat{a}^+ \hat{b} + \hat{b}^+ \hat{a}) , \]
where \( \hat{a} \) and \( \hat{b} \) satisfy the bosonic commutation relationship: \([\hat{a}, \hat{a}^+] = [\hat{b}, \hat{b}^+] = 1 \) and other combinations give 0.

A. Show that the Bogoliubov transformation of the form
\[ a^+ = uc^+ + vd \]
\[ b^+ = ud^+ + vc \]
is unitary if \( u^2 - v^2 = 1 \).
(Hint: show that \([c, c^+] = [d, d^+] = 1 \), and other combinations give 0.)

B. Show that in the new basis, the Hamiltonian can be reduced to
\[ H = \varepsilon (c^+ c + d^+ d) + \varepsilon - \alpha , \]
where
\[ \varepsilon = \sqrt{\alpha^2 - \beta^2} . \]

C. The energy of fundamental excitations of a BEC satisfy
\[ H = \frac{gn}{2} N + \frac{1}{2} \sum_{k=0}^{\infty} \left( \frac{\hbar^2 k^2}{2m} + gn_0 \right) (a_k^+ a_k + a_{-k}^+ a_{-k}) + gn_0 (a_k^+ a_{-k} + a_{-k}^+ a_k) \]
Use the result of C and construct the quasi-particle operators \( b_k, b_{-k} \) from \( a_k, a_{-k}, a_k^+, a_{-k}^+ \) that diagonalize the Hamiltonian.

D. Show that in the limit of large excited state population, the above treatment reduces to diagonalizing the 2x2 matrix we did on Monday 10/9/2017 in class. In what way is the 2nd order quantization method more fundamental?

Solution:
A. Algebra skipped
B. After the substitution, we have \( H = \ldots (c^+ c + d^+ d) + \ldots (cd + d^+ c^+) + \ldots \). To remove the off-diagonal term, we have \( 2\alpha uv + \beta (u^2 + v^2) = 0 \). Together with \( u^2 - v^2 = 1 \), we can use \( u^2 = \frac{1}{2} (\frac{\alpha}{\varepsilon} + 1 \) and \( v^2 = \frac{1}{2} (\frac{\alpha}{\varepsilon} - 1 \) to satisfy both equations. Here \( \varepsilon = \sqrt{\alpha^2 - \beta^2} \) is the desired eigenvalue.

C. Apply \( \alpha = \frac{\hbar^2 k^2}{2m} + gn_0 \) and \( \beta = gn_0 \)

D. Linearization of GP equation is the same as the Bogoliubov transformation for large phonon population \( a_k^+ a_{-k} >> 1 \). 2nd order quantization method handles vacuum and states with low occupancy number properly. Spontaneous emission and quantum fluctuations (the constant energy term \( \varepsilon - \alpha \) in vacuum) are good examples.

2. Can Bose-Einstein condensates be bound by gravity?
Gross-Pitaevskii equation describes the wavefunction of a Bose condensate at $T=0$:

$$\mu \Psi(r) = \left( \frac{\nabla^2}{2m} + V(r) + \frac{4\pi a}{m} |\Psi(r)|^2 \right) \Psi(r).$$

A. Argue that if one slowly reduces the trapping potential to zero $V(r) \to 0$, the sample with repulsive interaction $g>0$ will expand to infinity.

B. Will the above be true if we have so many atoms such that their gravitational attraction $F = -\nabla V_G$ can stabilize the expansion of the condensate?

(Hint: You can calculate or estimate using mean-field theory and $\nabla^2 V_G(x) = 4\pi G n(x)$.)

C. If your answer is yes, give a rough estimate on the mean density and the density profile of such a “Bose star” given $N$ atoms with scattering length $a$ in free space.

(Hint: You may simplify your calculation using Thomas-Fermi approximation.)

Solution:

A. Chemical potential is positive for positive density, so the cloud will expand.

B. Yes. If the condensate has $N$ particles in volume $L^3$.

Kinetic: $1/L^2$

Int.: $N/L^3$

Gravity: $-N/L$

So gravity can stabilize the expansion when the condensate expands too much.

C. When a bound state forms $\mu = 0$ and $|\Psi(r)|^2 = \frac{n(r)}{m}$, where $n(r)$ is the mass density.

We have $mV_G(r) + \frac{4\pi a}{m^2} n(r) = 0$ and thus $\nabla^2 n + \frac{Gm^3}{ah^2} n = 0$

The spherical symmetric solution of the above Helmholtz equation gives us the profile as

$$n(r < R) = \frac{A}{r} \sin \frac{\pi}{R} r \quad \text{and} \quad n(r > R) = 0,$$

where the radius of the star is $R = \frac{\pi a}{mG} m \sqrt{\frac{a}{mG}}$ and $A$ is a normalization constant. The density is estimated as $\bar{n} \sim N m R^3$.

Remark 1: For Cs atoms with scattering length = 1 Bohr, we have $R=30$ km! No one yet has a vacuum chamber this big. This result is sound for small particle numbers until we reach the Planck mass and they form black holes. So our labs are safe.

Remark 2: This problem is also a variation of the Thomas-Fermi model (1927), one of the most famous works by Enrico Fermi that started nuclear physics.