Physics 452 - Quantum Optics and Quantum Gasses Cheng Chin

HOMEWORK III (Due: 10/23/2017)

1. Eigenvalue-eigenfunction vs. 2nd quantized form

Consider a Hamiltonian given by

$\hat{H} = \alpha(\hat{a}^+\hat{a} + \hat{b}^+\hat{b}) + \beta(\hat{a}^+\hat{b}^+ + \hat{b}\hat{a})$, where $\hat{a}$ and $\hat{b}$ satisfy the bosonic commutation relationship: $[\hat{a},\hat{a}^+] = [\hat{b},\hat{b}^+] = 1$ and other combinations give 0.

A. Show that the Bogoliubov transformation of the form

$$a^+ = uc^+ + vd$$
$$b^+ = ud^+ + vc$$

is unitary if $u^2 - v^2 = 1$.

(Hint: show that $[c,c^+] = [d,d^+] = 1$, and other combinations give 0.)

B. Show that in the new basis, the Hamiltonian can be reduced to

$H = \varepsilon(c^+c + d^+d) + \varepsilon - \alpha$, where

$\varepsilon = \sqrt{\alpha^2 - \beta^2}$.

C. The energy of fundamental excitations of a BEC satisfy

$$H = \frac{gn}{2}N + \frac{1}{2}\sum_{k\neq 0}(\frac{\hbar^2k^2}{2m} + gn_0)(a_k^+a_k + a_{-k}^+a_{-k}) + gn_0(a_k^+a_{-k}^+ + a_{-k}a_{-k})$$

Use the result of C and construct the quasi-particle operators b_k, b_{-k} from $a_k, a_{-k}, a_k^+, a_{-k}^+$ that diagonalize the Hamiltonian.

D. Show that in the limit of large excited state population, the above treatment reduces to diagonalizing the 2x2 matrix we did on Monday 10/9/2017 in class. In what way is the 2nd order quantization method more fundamental?

Solution:

A. Algebra skipped

B. After the substitution, we have $H = ...(c^+c + d^+d) + ...(cd + d^+c^+) + ...$. To remove the the off-diagonal term, we have $2\alpha uv + \beta(u^2 + v^2) = 0$. Together with $u^2 - v^2 = 1$, we can use $u^2 = \frac{1}{2}(\frac{\alpha}{\varepsilon} + 1)$ and $v^2 = \frac{1}{2}(\frac{\alpha}{\varepsilon} - 1)$ to satisfy both equations. Here $\varepsilon = \sqrt{\alpha^2 - \beta^2}$ is the desired eigenvalue.

C**.** Apply $\alpha = \frac{\hbar^2k^2}{2m} + gn_0$ and $\beta = gn_0$

D. Linearization of GP equation is the same as the Bogoliubov transformation for large phonon population $a_k^+a_{-k} >> 1$. 2nd order quantization method handles vacuum and states with low occupancy number properly. Spontaneous emission and quantum fluctuations (the constant energy term $\varepsilon - \alpha$ in vacuum) are good examples.

2. Can Bose-Einstein condensates be bound by gravity?

Gross-Pitaevskii equation describes the wavefunction of a Bose condensate at T=0:

$$\mu\Psi(r) = (\frac{p^2}{2m} + V(r) + \frac{4\pi a\hbar^2}{m} |\Psi(r)|^2)\Psi(r).$$

A. Argue that if one slowly reduces the trapping potential to zero *V(r)*$\rightarrow$*0*, the sample with repulsive interaction g>0 will expand to infinity.

B. Will the above be true if we have so many atoms such that their gravitational attraction $F = -\nabla V_G$ can stabilize the expansion of the condensate?

(Hint: You can calculate or estimate using mean-field theory and $\nabla^2 V_G(x) = 4\pi G n(x)$.)

C. If your answer is yes, give a rough estimate on the mean density and the density profile of such a "Bose star" given N atoms with scattering length a in free space.

(Hint: You may simplify your calculation using Thomas-Fermi approximation.)

Solution:

A. Chemical potential is positive for positive density, so the cloud will expand.

B. Yes. If the condensate has N particles in volume L^3.

Kinetic: $1/L^2$

Int.: N/L^3

Gravity: $-N/L$

So gravity can stabilize the expansion when the condensate expands too much.

C. When a bound state forms $\mu = 0$ and $|\Psi(r)|^2 = \frac{n(r)}{m}$, where $n(r)$ is the mass density.

We have $mV_G(r) + \frac{4\pi a\hbar^2}{m^2} n(r) = 0$ and thus $\nabla^2 n + \frac{Gm^3}{a\hbar^2} n = 0$

The spherical symmetric solution of the above Helmholtz equation gives us the profile as $n(r < R) = \frac{A}{r} \sin\frac{\pi}{R} r$ and $n(r > R) = 0$, where the radius of the star is $R = \pi \frac{\hbar}{m}\sqrt{\frac{a}{mG}}$ and A is a normalization constant. The density is estimated as $\bar{n} \sim NmR^{-3}$.

Remark 1: For Cs atoms with scattering length = 1 Bohr, we have R=30 km! No one yet has a vacuum chamber this big. This result is sound for small particle numbers until we reach the Planck mass and they form black holes. So our labs are safe.

Remark 2: This problem is also a variation of the Thomas-Fermi model (1927), one of the most famous works by Enrico Fermi that started nuclear physics.