Physics 452 - Quantum Optics and Quantum Gasses Cheng Chin

HOMEWORK III (Due: 10/23/2017)

1. Eigenvalue-eigenfunction vs. 2nd quantized form

Consider a Hamiltonian given by

$\hat{H}=\alpha(\hat{a}^{+}\hat{a}+\hat{b}^{+}\hat{b})+\beta(\hat{a}^{+}\hat{b}^{+}+\hat{b}\hat{a})$, where $\hat{a}$ and $\hat{b}$ satisfy the bosonic commutation relationship: $[\hat{a},\hat{a}^{+}]=[\hat{b},\hat{b}^{+}]=1$ and other combinations give 0.

A. Show that the Bogoliubov transformation of the form

$$a^{+}=uc^{+}+vd$$
$$b^{+}=ud^{+}+vc$$

is unitary if $u^2-v^2=1$.

(Hint: show that $[c,c^{+}]=[d,d^{+}]=1$, and other combinations give 0.)

B. Show that in the new basis, the Hamiltonian can be reduced to

$H=\varepsilon(c^{+}c+d^{+}d)+\varepsilon-\alpha$, where

$\varepsilon=\sqrt{\alpha^2-\beta^2}$.

C. The energy of fundamental excitations of a BEC satisfy

$$H=\frac{gn}{2}N+\frac{1}{2}\sum_{k\neq 0}\left(\frac{\hbar^2k^2}{2m}+gn_0\right)(a_k^{+}a_k+a_{-k}^{+}a_{-k})+gn_0(a_k^{+}a_{-k}^{+}+a_{-k}a_{-k})$$

Use the result of C and construct the quasi-particle operators b_k, b_{-k} from a_k, a_{-k}, a_k^{+}, a_{-k}^{+} that diagonalize the Hamiltonian.

D. Show that in the limit of large excited state population, the above treatment reduces to diagonalizing the 2x2 matrix we did on Monday 10/9/2017 in class. In what way is the 2nd order quantization method more fundamental?

2. Can Bose-Einstein condensates be bound by gravity?

Gross-Pitaevskii equation describes the wavefunction of a Bose condensate at *T*=0:

$$\mu\Psi(r)=\left(\frac{p^2}{2m}+V(r)+\frac{4\pi a\hbar^2}{m}|\Psi(r)|^2\right)\Psi(r).$$

A. Argue that if one slowly reduces the trapping potential to zero $V(r)\to 0$, the sample with repulsive interaction $g>0$ will expand to infinity.

B. Will the above be true if we have so many atoms such that their gravitational attraction $F=-\nabla V_G$ can stabilize the expansion of the condensate?

(Hint: You can calculate or estimate using mean-field theory and $\nabla^2 V_G(x)=4\pi G n(x)$.)

C. If your answer is yes, give a rough estimate on the mean density and the density profile of such a "Bose star" given N atoms with scattering length a in free space.

(Hint: You may simplify your calculation using Thomas-Fermi approximation.)