Physics 452 - Quantum Optics and Quantum Gasses

HOMEWORK V (Due: 11/13/2017)

1. Math

Given Pauli matrices $\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, show that

A. $\lfloor \sigma_x, \sigma_y \rfloor = 2i\sigma_z$

B. $\exp(ia\sigma_z) = \cos a + \sigma_z \sin a$

C. $\exp(ia\sigma_z + ib\sigma_x) = \cos\sqrt{a^2+b^2}\,\hat{1} + \frac{a\sigma_z + b\sigma_x}{\sqrt{a^2+b^2}} \sin\sqrt{a^2+b^2}$

2. Density matrix and radiative cross sections

Consider a two-level atom interacting with an external radiation field, described by Hamiltonian $H = \frac{\hbar\omega_0}{2}\sigma_z + \frac{\hbar\Omega}{2}\sigma_x \cos\omega t$, where $\hbar\omega_0$ is the energy splitting of the atom between the ground and excited state, $\hbar\omega$ is the energy of a photon, Ω is the Rabi frequency. The wavefunction of the atom is $\psi = c_1 |1> + c_2 |2>$, where $|1>$ is the excited state and $|2>$ is the ground state.

A. Show that for small laser detuning $\Delta = \omega - \omega_0$, you can introduce a new basis $\psi_R = U\psi$ in the rotating frame and the Hamiltonian under rotating wave approximation (RWA) is given by

$H_R = UHU^{-1} \approx -\frac{\hbar\Delta}{2}\sigma_z + \frac{\hbar\Omega}{2}\sigma_x$

B. Show that the evolution of the density matrix $\rho_{ij} = <i|\psi_R><\psi_R|j>$ satisfy

$\dot{\rho}_{11} = \frac{i\Omega}{2}(\rho_{12} - \rho_{21})$, $\dot{\rho}_{12} = i\Delta\rho_{12} + \frac{i\Omega}{2}(\rho_{12} - \rho_{22})$, $\dot{\rho}_{22} = -\dot{\rho}_{11}$ and $\dot{\rho}_{21} = \dot{\rho}_{12}^*$.

C. Determine the duration t_π of a π-pulse that exactly flips the populations in the ground and excited states.

D. By introducing the decay of the excited state population, we rewrite the equations as $\dot{\rho}_{11} = \frac{i\Omega}{2}(\rho_{12} - \rho_{21}) - \Gamma\rho_{11}$ and $\dot{\rho}_{12} = i\Delta\rho_{12} + \frac{i\Omega}{2}(\rho_{12} - \rho_{22}) - \frac{\Gamma}{2}\rho_{12}$, show that in stationary state, we can write the solution as $\rho_{11} = \frac{1}{2}\frac{P}{1+P}$, where $P \equiv \frac{I/I_s}{1+4\Delta^2/\Gamma^2}$ is the saturation parameter and $\frac{I}{I_s} \equiv \frac{2\Omega^2}{\Gamma^2}$ defines the saturation intensity I_s.

C. Argue that the photon scattering rate of an atom is $s = \Gamma\rho_{11}$ and the atomic scattering cross section (the effective area of the shadow cast by an atom) is $\sigma = \frac{s\hbar\omega}{I}$.

D. Given that the maximum radiative cross section is $\sigma_{\max} = \frac{3\lambda^2}{2\pi}$ from atomic physics, show that the saturation intensity can be predicted from the linewidth as $I_s = \frac{\pi}{3}\frac{\Gamma\hbar\omega^3}{c^2}$.

E. In the presence of the linewidth Γ, how should you do if you wish to apply a π-pulse to transfer at least 90% of the ground state population to the excited state? Given an estimate how high the intensity you will need to accomplish this.

3. Optical pumping

Alfred Kastler won the Nobel Prize in 1966 for the invention of optical pumping, which polarize atoms in a specific internal state. Here we try to develop a simple model for optical pumping based on what we have learned.

Assume that in addition to the two levels that we have considered in 3, there is a second ground state $|3>$ that is not excited by the external radiation (typically due to selection rules). Excited population ρ_{11}, however, can decay to both $|2>$ and $|3>$ with probability $1-\varepsilon$ and ε. Argue that the density matrix should be written as

$$\dot{\rho}_{11} = \frac{i\Omega}{2}(\rho_{12}-\rho_{21}) - \Gamma\rho_{11}$$

$$\dot{\rho}_{22} = -\frac{i\Omega}{2}(\rho_{12}-\rho_{21}) + \Gamma(1-\varepsilon)\rho_{11}$$

$$\dot{\rho}_{33} = \Gamma\varepsilon\rho_{11}$$

$$\dot{\rho}_{12} = i\Delta\rho_{12} + \frac{i\Omega}{2}(\rho_{12}-\rho_{22}) - \frac{\Gamma}{2}(1-\varepsilon)\rho_{12} = \dot{\rho}_{21} \text{ and}$$

$$\rho_{23} = \rho_{32} = \rho_{13} = \rho_{31} = 0$$

A. Show that after long evolution time, the stationary solution is $\rho_{33} = 1$ and $\rho_{11} = \rho_{22} = 0$.

B. If we start with all atoms in $\rho_{22} = 1$ and assume $\varepsilon << 1$, calculate/estimate the growth of the population $\rho_{33}(t)$. How fast does the pumping efficiency reach 50%, namely, $\rho_{33} = 0.5$?

4. Ramsey spectroscopy

Define Bloch vector of a spinor as $\vec{v} = tr[\vec{\sigma}\rho]$, where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is given by the Pauli matrices. Given the RWA, the spinor evolves according to $i\partial_t = -\Delta\sigma_z + \Omega\sigma_x$.

Show that

A. $\rho = \frac{1}{2}(\hat{1} + \vec{v}\cdot\vec{\sigma})$

B. The evolution operator $U = e^{t\partial_t}$ rotates the Bloch vector around the axis $(\Omega, 0, -\Delta)$.

C. Ramsey sequence goes as follows:

- first optical pump all atoms to an initial state |0>
- apply a π/2-pulse in $t_\pi / 2$.
- let the system evolve for time T, and
- a second π/2-pulse.

Calculate the population after the sequence. And plot the population ρ_{11} as a function of the detuning of the laser $\Delta = \omega - \omega_R$.

D. If you apply the above sequence to an ensemble of *N* atoms and immediately measure the atomic population ρ_{ii} , how would you use the result to precisely determine the atomic energy splitting? Show that the uncertainty of the measurement would be consistent with the single particle uncertainty principle $\delta\omega = c\frac{1}{t\sqrt{N}}$, where *c* is a numerical coefficient.

Hint: you may use the result from 1.