Physics 452 - Quantum Optics and Quantum Gasses
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HOMEWORK VI (Due: 11/20/2017)

1. Mollow triplets
In the dress atom calculation, we show that the photons scattered by the atoms can have three components, called Mollow triplets.

A. Given atomic energy $H_a = \frac{\hbar\omega_0}{2}\sigma_z$, radiation field energy $H_p = \hbar\omega b^+ b$, detuning $\Delta = \omega - \omega_0$, and the atom-photon coupling under RWA $H_{ap} = \frac{g}{2}(\sigma^+ b + \sigma b^+)$, show that the eigenstates are $|1>=\cos\theta|g,n+1>+\sin\theta|e,n>$ and $|2>=-\sin\theta|g,n+1>+\cos\theta|e,n>$, where $\tan 2\theta = g^2 n/\hbar\Delta$, $n>>1$, the eigen-energies are $E_1 = (n+1)\hbar\omega - \frac{\hbar\Delta}{2} + \frac{\hbar\Delta'}{2}$ and $E_2 = (n+1)\hbar\omega - \frac{\hbar\Delta}{2} - \frac{\hbar\Delta'}{2}$, $\hbar\Delta' = \sqrt{\Delta^2 + \Omega^2}$ and $\Omega = g\sqrt{n}$ is the Rabi frequency.

B. Show that the scattered photons will have 3 spectral lines at ω_0 and $\omega_0 \pm \Delta'$.

C. Treat spontaneous emission semi-classically as decay process that transfers population in state $|e,n>$ to state $|g,n>$ with a rate Γ. Show that the equilibrium population in the eigen-state $|1>$ and $|2>$ are $\rho_{11} = \frac{\cos^4\theta}{\sin^4\theta + \cos^4\theta}$ and $\rho_{22} = \frac{\sin^4\theta}{\sin^4\theta + \cos^4\theta}$, respectively.

D. Determine the scattering rates into the 3 components. Comment on how the spectral strength of the 3 lines increases as the intensity of the radiation field $I_p \propto <b^+ b>$.

Solution:
A. Diagonalize the 2x2 Hamiltonian in the space of $|g,n+1>$ and $|e,n>$.
B.

$\omega - \Delta'$ component: $\Gamma_{12}\rho_{22} = \Gamma\cos^4\theta \frac{\sin^4\theta}{\sin^4\theta + \cos^4\theta} = \Gamma\frac{\sin^4\theta\cos^4\theta}{\sin^4\theta + \cos^4\theta}$

ω component: $\Gamma_{11}\rho_{11} + \Gamma_{22}\rho_{22} = \Gamma\frac{\sin^2\theta\cos^6\theta + \sin^6\theta\cos^2\theta}{\sin^4\theta + \cos^4\theta} = \Gamma\sin^2\theta\cos^2\theta$

$\omega - \Delta'$ component: $\Gamma_{21}\rho_{11} = \Gamma\sin^4\theta \frac{\cos^4\theta}{\sin^4\theta + \cos^4\theta} = \Gamma\frac{\sin^4\theta\cos^4\theta}{\sin^4\theta + \cos^4\theta}$

C. We may consider weak intensity limit. Scattering rate ~ I ~ θ^2. Only the ω component has the θ^2 dependence, which means it is a single photon process. The side peaks comes from two-photo process.

2. Dipole force and optical potential revisited

Based on the dressed atom picture and the results in 1 we will derive the more accurate formula for the light shift.

Consider an inhomogeneous radiation intensity $I(x)$. Light shift of an atom is the energy needed to move it adiabatically (slowly enough) from infinity where $I(\infty)=0$ toward the laser beam.

A. At position x, argue that the radiation force can be written as

$$F(x) = -\frac{\hbar\nabla\Delta'(x)}{2}[\rho_{11}(x)-\rho_{22}(x)]$$

B. Show that the optical force comes from an optical potential $U(x)$ given by

$$U = \frac{\hbar\Delta}{2}\ln(1+\frac{I/I_{sat}}{4\Delta^2/\Gamma^2})$$

C. Show that this result is consistent with our calculation based on semi-classical treatment in the limit of low intensity $I << I_{sat}$ or large detuning $\Delta >> \Gamma$.

$$U = \frac{\hbar\Gamma}{8}\frac{I/I_{sat}}{\Delta/\Gamma}$$

D. Comment on the difference between the light shift we derive here and the one from the Lorentz model: $\Delta E = -\mathrm{Re}\,\alpha_{AC}\frac{I}{2}$, where α_{AC} is the AC polarizability. Show that the difference occurs mostly near the resonance.

Hint: $I/I_{sat} = 2\Omega^2/\Gamma^2$.

References:

J.P. Gordon and A. Ashkin, PRA 21 1606 (1980)
J. Dalibard and C. Cohen-Tannoudji, J. Opt. Soc Am. B 2 1707 (1985)

Solution:

A. Use <F>=- d<E(x)>/dx, where <E(x)> is the expectation value of the energy shift.

B. Integrate the result in A.

C. Expand the result in B in the limit of small $\frac{I/I_{sat}}{4\Delta^2/\Gamma^2}$.

D. At an intensity I, light shift approaches zero according to $U \approx \frac{\hbar\Delta}{2}\ln\frac{I/I_{sat}}{4\Delta^2/\Gamma^2}$ for small detuning $|\Delta| << \Gamma$. Lorentz model gives $U \approx \frac{I}{2\omega\beta^2}\Delta$, which has a different dependence on Δ and *I*.

3. Electromagnetically induced transparency

Consider an ensemble of identical atoms whose dynamics can be described by taking into account only three of its eigenstates: $|1>$, $|2>$, and $|3>$. In the absence of electromagnetic fields, all atoms are assumed to be in the lowest energy state $|1>$. State $|2>$ has the same parity as $|1>$ and is assumed to have a very long coherence time. The highest energy state $|3>$ is of opposite parity and has a non-zero electric-dipole coupling to both $|1>$ and $|2>$. A resonant or near resonant non-perturbative radiation field of frequency ω_C, termed the control field, is applied on the $|2>$ to $|3>$ transition, and a probe field of frequency ω_P is applied on the $|1>$ to $|3>$ transition.

EIT is primary concerned with the modification of the linear and non-linear properties of the typically perturbative probe field.

A. Show that under a proper rotating wave approximation, the Hamiltonian can be written as

$H_{\text{int}} = -\Delta_P |3><3| - \Delta_C |2><2| - (\Omega_P |1><3| + \Omega_C |2><3| + h.c.)$,

where $\Delta_P = \omega_P - \omega_{13}$, $\Delta_C = \omega_C - \omega_{23}$, and Ω_P and Ω_C are the Rabi frequencies of the control and probe fields.

B. Given the wavefunction $|\psi> = c_1|1> + c_2|2> + c_3|3>$, show that the equation of motion is given by

$$\dot{c}_1 = i\Omega^*_P c_3$$
$$\dot{c}_2 = i(\Delta_P - \Delta_C)c_2 + i\Omega^*_C c_3$$
$$\dot{c}_3 = -(\frac{\Gamma}{2} - i\Delta_P)c_3 + i\Omega_P c_1 + i\Omega_C c_2,$$

where Γ is the decay rate of $|3>$.

C. Argue that the sample is transparent to the probe beam when $\operatorname{Im}\rho_{13} = 0$. Show that such condition can be achieved when the two radiation fields are detuned by the same amount $\Delta_P = \Delta_C$.

Hint: For simplicity, you may assume the probe is weak enough such that $c_1 \approx 1$.

D. For the special case $\Delta_P = \Delta_C = 0$, plot the absorption of the probe beam as a function of its frequency with no, weak and strong control beam.

Solution

A. Extending the result in HW5 #2, we can introduce the new basis $|1> = e^{i\omega_p t}|1>_{old}$ and $|2> = e^{i\omega_C t}|2>_{old}$ and offset the energy of state 1 to zero.

B. You may derive the result using Schroedinger's equation with RWA.

C. As we learned from the Lorentz model, the imaginary part leads to energy dissipation. Thus a perfect transparency for the probe beam requires $\operatorname{Im}\rho_{13} = 0$.

In equilibrium, we have $(\Delta_P - \Delta_C)c_2 + \Omega^*_C c_3 = 0$ and $i\Omega_P c_1 = (\frac{\Gamma}{2} - i\Delta_P)c_3 - i\Omega_C c_2$.

Eliminating c_2 and multiplying c_3^*, we get $c_1 c_3^* = \rho^*_{13} = |c_3|^2 \left[\frac{|\Omega_C|^2}{\Omega_P(\Delta_p - \Delta_C)} - \frac{\Delta_p - i\Gamma/2}{\Omega_P} \right]$,

whose imaginary party vanishes when $\Delta_p = \Delta_C$.

D. When the control field is on resonance we have $\rho_{13} = \frac{|c_3|^2}{\Omega_P}(\frac{|\Omega_C|^2}{\Delta_p} - \Delta_p - i\Gamma/2)$.

Providing a proper normalization, we can write the absorption as

$$\alpha \propto \mathrm{Im}\rho_{13} \propto \frac{\Gamma/2}{(\frac{|\Omega_C|^2}{\Delta_p} - \Delta_p)^2 + \frac{\Gamma^2}{4}} = \frac{\Gamma}{2}\frac{1}{1+(1-\frac{|\Omega_C|^2}{\Delta_p^2})^2\frac{4\Delta_p^2}{\Gamma^2}} .$$

Note that the above expression recovers the Lorentzian absorption for a two-level system $\alpha \propto s = \frac{\Gamma}{2}\frac{1}{1+\frac{4\Delta_p^2}{\Gamma^2}}$ when the control beam is off $\Omega_C = 0$ and the probe beam is weak.

When the control beam is on, absorption at the resonance is fully suppressed and such transparency window has a spectral width of $\frac{|\Omega_C|^2}{\Gamma} \propto I_C$, which is proportional to the control beam intensity. This is the EIT.

Very close to the resonance (or equivalently very strong control beam), we obtain a quadratic dependence on the detuning $\alpha \propto \frac{\Gamma}{2}\frac{\Delta_p^2}{1+\frac{4|\Omega_C|^4}{\Gamma^2}}$.