Physics 452 - Quantum Optics and Quantum Gasses
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HOMEWORK VI (Due: 11/20/2017)

1. Mollow triplets
In the dressed atom picture calculation, we show that the light scattered by the atoms can have three spectral components, called Mollow triplets. Here we will determine their frequencies and strength.

A. Given the atomic energy $H_a = \frac{\hbar\omega_0}{2}\sigma_z$, radiation field energy $H_p = \hbar\omega b^+ b >> \hbar\omega$, detuning $\Delta = \omega - \omega_0$, and the atom-photon coupling under RWA: $H_{ap} = \frac{g}{2}(\sigma^+ b + \sigma b^+)$, show that the eigenstates are $|1>=\cos\theta|g,n+1>+\sin\theta|e,n>$ and $|2>=-\sin\theta|g,n+1>+\cos\theta|e,n>$, where $\tan 2\theta = g^2 n/\hbar\Delta$, the eigen-energies are $E_1 = (n+1)\hbar\omega - \frac{\hbar\Delta}{2} + \frac{\hbar\Delta'}{2}$ and $E_2 = (n+1)\hbar\omega - \frac{\hbar\Delta}{2} - \frac{\hbar\Delta'}{2}$, $\hbar\Delta' = \sqrt{\Delta^2 + \Omega^2}$ and $\Omega = g\sqrt{n}$ is the Rabi frequency.

B. Show that the scattered photons have 3 frequencies at ω_0 and $\omega_0 \pm \Delta'$.

C. Treat spontaneous emission semi-classically as decay process that transfers population from state $|e,n>$ to state $|g,n>$ with a rate constant Γ and $n \approx n+1 >> 1$. Show that the equilibrium populations in the eigen-states $|1>$ and $|2>$ are $\rho_{11} = \frac{\cos^4\theta}{\sin^4\theta + \cos^4\theta}$ and $\rho_{22} = \frac{\sin^4\theta}{\sin^4\theta + \cos^4\theta}$, respectively.

D. Determine the scattering rates into the 3 components. Comment on how the spectral strengths scale with the intensity of the radiation field $I_p \propto < b^+ b >$.

2. Dipole force and optical potential revisited
Based on the dressed atom picture we discussed, and the results in problem 1 we are now in the position to derive the right formula for the light shift.
Consider an inhomogeneous radiation intensity $I(x)$. Light shift of an atom is the energy needed to move it adiabatically (slowly enough) from infinity where there is no light $I(\infty) = 0$ into the radiation field.

A. At position x, show/argue that the radiation force on the atoms is given by

$$F(x) = -\frac{\hbar\nabla\Delta'(x)}{2}[\rho_{11}(x) - \rho_{22}(x)]$$

B. Show that the optical force is effectively a result of the optical potential

$$U = \frac{\hbar\Delta}{2}\ln\left(1 + \frac{I/I_{sat}}{4\Delta^2/\Gamma^2}\right).$$

(Hint: $I/I_{sat} = 2\Omega^2/\Gamma^2$.)
Show that this result is consistent with our calculation based on semi-classical treatment in the limit of low intensity $I << I_{sat}$ or large detuning $\Delta >> \Gamma$.

$$U = \frac{\hbar\Gamma}{8}\frac{I/I_{sat}}{\Delta/\Gamma}$$

C. Comment on the difference between the light shift we derive here and the one from the Lorentz model: $\Delta E = -\mathrm{Re}\,\alpha_{AC}\frac{I}{2}$, where α_{AC} is the AC polarizability. Show that the difference matters near the resonance.

References:
J.P. Gordon and A. Ashkin, PRA 21 1606 (1980)
J. Dalibard and C. Cohen-Tannoudji, J. Opt. Soc Am. B 2 1707 (1985)

3. Electromagnetically induced transparency (EIT)
Consider an ensemble of identical atoms whose dynamics can be described by taking into account only three of its eigenstates: $|1>$, $|2>$, and $|3>$. In the absence of electromagnetic fields, all atoms are assumed to be in the lowest energy state $|1>$. State $|2>$ has the same parity as $|1>$ and is assumed to have a very long coherence time. The highest energy state $|3>$ is of opposite parity and has a non-zero electric-dipole coupling to ground states. A resonant or near resonant non-perturbative radiation field of frequency ω_C, termed the control field, is applied on the $|2>$ to $|3>$ transition, and a probe field of frequency ω_P is applied on the $|1>$ to $|3>$ transition.

EIT is primary concerned with the modification of the linear and non-linear response of the typically perturbative probe field.
A. Show that under a proper rotating wave approximation, the Hamiltonian can be written as

$$H_{\mathrm{int}} = -\Delta_P |3><3| - \Delta_C |2><2| - (\Omega_P |1><3| + \Omega_C |2><3| + h.c.),$$

where $\Delta_P = \omega_P - \omega_{13}$, $\Delta_C = \omega_C - \omega_{23}$, and Ω_P and Ω_C are the Rabi frequencies of the control and probe fields.
B. Given the wavefunction $|\psi> = c_1|1> + c_2|2> + c_3|3>$, show that the equation of motion is given by

$$\dot{c}_1 = i\Omega_P^* c_3$$

$$\dot{c}_2 = i(\Delta_P - \Delta_C)c_2 + i\Omega_C^* c_3$$

$$\dot{c}_3 = -(\frac{\Gamma}{2} - i\Delta_P)c_3 + i\Omega_P c_1 + i\Omega_C c_2,$$

where Γ is the decay rate of $|3>$.
C. Argue that the sample is transparent to the probe beam when $\mathrm{Im}\,\rho_{13} = 0$. Show that such condition can be achieved when the two radiation fields are detuned by the same amount $\Delta_P = \Delta_C$.

Hint: For simplicity, you may assume the probe is weak enough such that $c_1 \approx 1$.

D. For the special case $\Delta_P = \Delta_C = 0$, plot the absorption of the probe beam as a function of its frequency with no, weak and strong control beam.