Physics 452 - Quantum Optics and Quantum Gasses

Midterm (Due: 11/06/2017)

1. Time-dependent dynamics of a Bose-Einstein condensate

We consider a Bose-Einstein condensate at zero temperature in a harmonic potential $V = \frac{1}{2}m(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)$ with time-dependent trap frequencies $\omega_i = \omega_i(t)$. The general time evolution of the condensate is described by the time-dependent Gross-Pitaevskii equation, $i\hbar\partial_t\psi = [E_k + V(r,t) + g|\psi|^2]\psi$, which in the case of harmonic trapping potentials allows for simple scaling solutions.

A. Assume the stationary density distribution follows the equilibrium Thomas-Fermi density distribution with radius in the *j*-th direction given by $R_j(t \le 0)$. Argue that the evolution of the gas for times $t > 0$ can be described as a "dilatation": any part of the cloud moves along the trajectory described by $R_j(t) = \lambda_j(t)R_j(0)$.

B. Argue based on classical mechanics or with GP equation that the time evolution of $\lambda_j(t)$ is governed by the following differential equation: $\ddot{\lambda}_j(t) = \frac{\omega_j^2(0)}{\lambda_j(t)\lambda_x(t)\lambda_y(t)\lambda_z(t)} - \omega_j^2(t)\lambda_j(t)$.

C. Show that the approximation solution of the time-dependent Gross-Pitaevskii equation for harmonic trapping potential is given by the scaling solution

$$n(r,t) = \frac{\mu - \sum_j \frac{1}{2}m\omega_j^2(0)r_j^2/\lambda_j^2(t)}{g\lambda_x(t)\lambda_y(t)\lambda_z(t)}$$

with the scaling parameters λ_j fulfilling the coupled equation in B.

D. Consider the following case: a Bose-Einstein condensate is initially prepared in a cylindrically symmetric trap at zero temperature with trapping frequencies $\omega_x = \omega_y \equiv \omega_r >> \omega_z$. At time t = 0 the trap is suddenly switched off, i.e. $\omega_r(t>0) = \omega_z(t>0) = 0$. This is equivalent to releasing the condensate in free space, and is called time-of-flight expansion. Determine $n(r,t)$.

Hint: Introduce the dimensionless time variable $\tau = \omega_r(0)t$ and the asymmetry parameter $\varepsilon = \omega_z(0)/\omega_r(0) << 1$. Expand the coupled differential equations in powers of ε. Find the solution for λ_z to zero order ε, and the solution for λ_r to second order in ε. The initial conditions are given by $\lambda_j(0) = 1$ and $\dot{\lambda}_j(0) = 0$

F. Sketch the evolution of the radial and axial sizes of the Bose-Einstein condensate after the release. Comment on its relationship with uncertainty principle.

Reference: Y. Castin and R. Dum, PRL 77, 5315, (1996)

2. Bose condensates in a spatial double well potential
Consider N>>1 atoms with repulsive interaction in a symmetric double-well potential. Depending on the height of the barrier, the system can exhibit very different behavior and characteristics. A high barrier would e.g. correspond to two independent BECs without exchange of particles between them, whereas a very small barrier would just correspond to a slight modification of the ground-state wavefunction of the BEC. Less trivial is the case where the ground-state wavefunction exhibits two localized parts with a non-zero tunneling probability. In this case one can think of two BECs with fluctuating particle number.

We label the single-particle ground state wavefunction in each well a and b by $\psi_a(r)$ and $\psi_b(r)$. To investigate quantum fluctuating variables we work in the second quantization form. We introduce the operators $\hat{a}$, $\hat{a}^+$ and $\hat{b}$, $\hat{b}^+$, corresponding to the annihilation and creation of a particle in the matter-wave modes ψ_a and ψ_b.

A. Show that the Bose-Hubbard Hamiltonian in second quantization form can be approximated by $\hat{H} = -t(\hat{a}^+\hat{b} + \hat{a}\hat{b}^+) + \frac{U}{2}(\hat{a}^+\hat{a}\hat{a}^+\hat{a} + \hat{b}^+\hat{b}\hat{b}^+\hat{b})$. Indicate the approximations you need to reach this result.

B. We consider first the effect of the tunneling term, without interactions: $t \neq 0$ and $U = 0$. Show that the Hamiltonian can be rewritten as $\hat{H} = -t(\hat{c}_+^+\hat{c}_+ - \hat{c}_-^+\hat{c}_-)$, where $\hat{c}_\pm = 2^{-1/2}(\hat{a} \pm \hat{b})$ is the annihilation operator of a particle in the mode $\psi_\pm = 2^{-1/2}(\psi_a \pm \psi_b)$. Show that for *N* particles the eigenstates are the states where the $\pm$ modes are filled with $N_\pm$ particles, labeled as $|N_+, N_->$, and that the ground state is $|g>=|N,0>$.

C. We are interested in the number of particles on each site in the ground state. To calculate this we have to work in the $|N_a, N_b>$ basis, corresponding to the filling of the modes ψ_a and ψ_b. Expand the ground state in this basis as $|g>=\sum_0^N A_i\,|N-i,i>$, using the definition of the creation operators $|g>=(N!)^{-1/2}(\hat{c}_+^+)^N\,|0>$. What is the probability distribution $p(N_a)$ of the atom numbers on the site *a* (or *b*). What are the mean values and variance of N_a and N_b?
D. Consider now the opposite limit $t = 0$ and $U \neq 0$. Determine the ground and first two excited states for both $U > 0$ and $U < 0$ as well as their energies.

3. Bose condensates in double well potential in the momentum space
Here we consider bosonic atoms that can (in principle) condense to more than 1 ground states. Starting from the total energy of the system (with finite density in a large box)

$$H = \sum_p \varepsilon_p a_p^+ a_p + \frac{g}{2} \sum_{p,p',q} a_{p+q}^+ a_{p'-q}^+ a_p a_{p'} ,$$

where $g > 0$ and the kinetic energy ε_p has a 2-fold degeneracy at it minimum values $\varepsilon_{\pm p_0} = 0$ at momenta $p = \pm p_0$. The exact form of ε_p is not important, but for concreteness, you can take $\varepsilon_p = \alpha(p^2 - p_0^2)^2$ and $\alpha > 0$. Such a kinetic energy has been observed for particles in a gauge field.

Since we have two possible ground states, how will *N>>1* bosons condense at low temperatures? We may consider 4 possible options:

a. all *N* particles in a single momentum state like $(\hat{a}_{\pm p_0}^+)^N | 0 >$
b. 50%-50% mixture like $(\hat{a}_{p_0}^+)^{N/2} (\hat{a}_{-p_0}^+)^{N/2} | 0 >$
c. a superposition state like $2^{-N/2} (\hat{a}_{p_0}^+ + \hat{a}_{-p_0}^+)^N | 0 >$ and
d. a cat state like $2^{-1/2} [(\hat{a}_{p_0}^+)^N + (\hat{a}_{-p_0}^+)^N] | 0 >$?

A. Show that the single momentum state and the cat state have lower energy than the other options. Can the Hamiltonian distinguish these 2 ground states?

Hint: evaluate <*H*>.

B. If we, however, expect that the total momentum is zero $< \sum_i p_i >= 0$ and cat state is out of the question (too delicate). Which of the following two states has a lower energy, b or c? Evaluate their total energies if you can, or offer a physical reason if the math is out of hand.

C. If we prepare exactly N/2 particles in the states with $p = \pm p_0$, it looks like the only choice for the condensate wavefunction is $(\hat{a}_{p_0}^+)^{N/2} (\hat{a}_{-p_0}^+)^{N/2} | 0 >$. Comment on its energy and can there be a different arrangement of the atoms with an even lower energy? If you know the ground state or at least a state with even lower energy, describe it mathematically or argue why the state you have even might has even lower energy.

4. Diverging scattering length at Feshbach resonance

One of the weirdest expectations near a Feshbach resonance is the diverging scattering length $a \to \pm\infty$. Given $a = a_{bg}(1 - \frac{\Delta}{B - B_0})$ and assume $a_{bg}, \Delta > 0$ for simplicity, we have strong repulsion when we are slightly below the resonance $B \le B_0$ and strong attraction slightly above the resonance $B \ge B_0$. How are these 2 scenarios connect each other exactly on Feshbach resonance $B = B_0$?

Here we perform a thought experiment by considering two atoms in a weak spherical harmonic potential . The Hamiltonian of the system can be written in the CM and relative coordinates as

$$H = \frac{P^2}{2M} + \frac{1}{2} M\omega^2 X^2 + \frac{p^2}{2\mu} + \frac{1}{2}\mu\omega^2 x^2 + V(r),$$

where we can use the pseudo potential $V(r) = \frac{4\pi a\hbar^2}{2\mu}\delta(r)\partial_r r$ to model their interaction.

Once we determine the ground state, our goal is to look at the atoms and see whether they attract each other or repel each other.

A. Without the interaction $a = 0$, both atoms are in the ground state of the harmonic potential. Argue that the two atoms behave independently. Show that if one atom is detected at a location $x=x_1>0$ (along the x-axis), it will not influence the expected location of the second particle at $\langle x_2 \rangle$, which is identically zero.

B. Now with a small scattering length $|a| << l$, where $l = \sqrt{\hbar / \mu\omega}$ is the length scale of the harmonic potential. You may treat the interaction as a perturbation to the atom pair. When the first atom is detected at $x=x_1$, what is the expected location of the second particle $\langle x_2 \rangle$? Does the result show that scattering length determines the interaction between the two atoms?

C. Now tune $a \to \pm\infty$ (equivalently $B = B_0$) and describe what you expect to see and show that there is actually no such abrupt change from strong repulsion to strong attraction. Comment on whether the two atoms repel or attract?

Hint: The pseudo potential $V(r) = \frac{4\pi a\hbar^2}{2\mu}\delta(r)\partial_r r$ essentially demands the boundary condition as $\lim_{r\to 0}\frac{r\psi}{(r\psi)'} = -a$. Thus on resonance, we have $\lim_{r\to 0}(r\psi)' = 0$.