P238 Atomic Physics 2018 Cheng Chin

Home Work 2

(Due: 11/1/2018)

1. Bloch vector $\vec{b} = (u, v, w)$
   Show that the following two definitions are equivalent
   a. Textbook (Foot) density matrix: $u = 2\,\mathrm{Re}[\rho_{12}]$, $v = 2\,\mathrm{Im}[\rho_{12}]$, and $w = \rho_{22} - \rho_{11}$.
   b. Class definition: $u =< \sigma_x >$, $v =< \sigma_y >$, and $w =< \sigma_z >$, where
   $(\sigma_x, \sigma_y, \sigma_z) = (\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix})$ are the Paul matrices

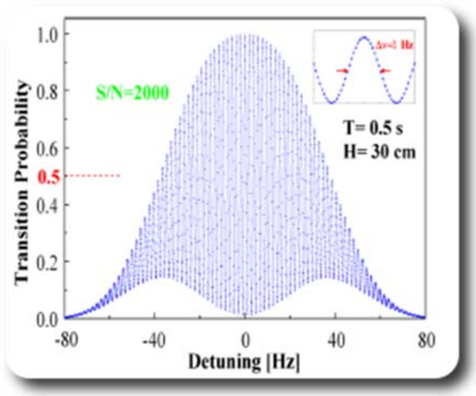
2. Evolution of wavefunction and $\pi/2$ and π pulses (again)
   A laser pulse of time duration t, interacts with an atom with
   $V = \hbar\left(\Omega_x \cos \omega t + \Omega_y \sin \omega t\right)(|2 >< 1| + |1 >< 2|)$

   A. Show that Schroedinger's equation in the rotating frame is $i\partial_t|\phi >= \frac{1}{2}\left(\sigma_x, \sigma_y, \sigma_z\right) \cdot \vec{\Omega}|\phi >$ and the evolution of the Bloch vector is $\partial_t \vec{\mathrm{b}} = \vec{\Omega} \times \vec{\mathrm{b}}$, where $\vec{\Omega} = \left(\Omega_x, \Omega_y, -\Delta\right)$.
   B. After the pulse, the atom is in the final state $|\phi(t) >= U(t)|\phi(0) >$, show that the evolution operation can be written a$U(t) = \hat{1}\cos\frac{\Omega t}{2} - i \sin\frac{\Omega t}{2}\left(\sigma_x, \sigma_y, \sigma_z\right) \cdot \frac{\vec{\Omega}}{\Omega}$, where $\Omega = |\vec{\Omega}| = \sqrt{{\Omega_x}^2 + {\Omega_x}^2 + \Delta^2}$ is the generalized Rabi frequency.

   C. Given $\Omega_y = 0$, show that a $\pi/2$ pulse with $\Delta = 0, t = \frac{\pi}{2\Omega}$ yields the evolution operator
   $$U_{\pi/2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} .$$
   D. Starting with an atom in the ground state $|\psi >= \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, show that the pulse rotate the Bloch vector from $\vec{b} = (0,0,1)$ by 90° along the x-axis.
   E. Show that if you apply 4 $\pi/2$ pulses to the system, you get $-|\psi >= \begin{bmatrix} 0 \\ -1 \end{bmatrix}$. The Bloch vector, however, goes back to $\vec{b} = (0,0,1)$.

3. Ramsey fringes

Here we will try to derive the analytic form of Ramsey fringes shown in Foot, Fig. 7.3 (page 134).



Here dots are the measurement and the solid line is the theory given in Eq. (7.53):

$$I = I_0 \cos^2(\frac{kd}{2}\sin\theta)\,\mathrm{sinc}^2(\frac{ka}{2}\sin\theta)\,.$$

Let's try to derive it.

Starting with an atom in the ground state $|\phi> = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, Ramsey fringes are obtained by

1. Applying a $\pi/2$ pulse with a frequency detuning Δ and $\Omega_y = 0$.
2. Waiting for time T.
3. Applying a second identical $\pi/2$ pulse.
4. Measuring the population in the excited state.

A. Derive the formula in the textbook.

B. Use the evolution operator from homework 2, $U(t) = \hat{1}\cos\frac{\Omega' t}{2} - i\sin\frac{\Omega' t}{2}(\frac{\Omega}{\Omega'}\hat{\sigma}_x + \frac{\Delta}{\Omega'}\hat{\sigma}_z)$,

C. Show that with T=0.5 s, you do get the indicated linewidth of 1Hz. What does H=30cm mean?

4. Equilibrium of atomic population in the ground and excited states

Using RWA and the rotating frame, we get Hamiltonian $H = -\frac{\Delta}{2}\sigma_z + \frac{\Omega}{2}\sigma_x$ and density matrix follows $i\hbar\partial_t \rho = [H, \rho]$.

A. Derive the explicit form of the equation of motion for $\rho_{ij} = <i|><|j>$, and show that they are (1 is for ground state, and 2 is for excited state)

$$\dot{\rho}_{22} = i\frac{\Omega}{2}(\rho_{21} - \rho_{12})$$

$$\dot{\rho}_{11} = i\frac{\Omega}{2}(\rho_{12} - \rho_{21})$$

$$\dot{\rho}_{12} = -i\Delta\rho_{12} + i\frac{\Omega}{2}(\rho_{22} - \rho_{11})$$

$$\dot{\rho}_{21} = i\Delta\rho_{21} - i\frac{\Omega}{2}(\rho_{22} - \rho_{11})$$

B. When the excited state decay is included, we add additional terms of $-\Gamma\rho_{22}$, $+\Gamma\rho_{22}$, $-\frac{\Gamma}{2}\rho_{12}$ and $-\frac{\Gamma}{2}\rho_{21}$ to the above 4 equations respectively. Justify the signs of the terms and why is there a factor of 1/2 in the latter 2 cases?
(Hint: Consider an atom initially in the excited state, no laser is applied. It will decay exponentially with the rate constant Γ. What should you do to the equations to obtain the expected decay behavior?)

C. Show that when the laser is on ($\Omega>0$), the excited state population eventually approaches an equilibrium value, given by

$$\rho_{22} = \frac{1}{2}\frac{p}{1+p},$$

where the saturation parameter is defined as $p = \frac{I/I_S}{1+4\Delta^2/\Gamma^2}$

and the laser intensity I and the saturation intensity I_S are related by $\frac{I}{I_S} = \frac{2\Omega^2}{\Gamma^2}$.

(Hint: it is equivalent to show that $\rho_{22} = \frac{1}{2}\frac{2\Omega^2/\Gamma^2}{1+4\Delta^2/\Gamma^2+2\Omega^2/\Gamma^2}$, but these quantities: intensity, saturation intensity, and saturation parameter are frequently used in the literature.)