P238Atomic Physics 2018 Cheng Chin

Home Work 3

(Due: 11/13/2018)

1. Einstein's thought on atomic linewidth and cross secton
While we said in the class that atomic linewidth and cross section depend on the atomic species and the specific transition. In the same year Einstein published the work on general relativity, he also considered the following problem.

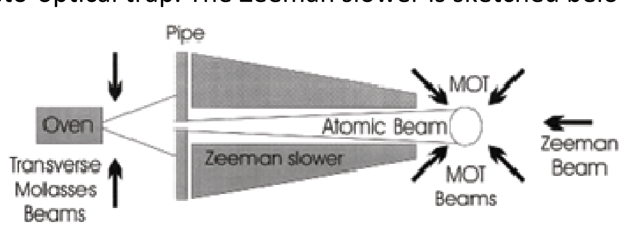
Consider an atom in a box at temperature T. In thermal equilibrium, atomic populations in the excited and ground state should follow the Boltzmann distribution $\frac{\rho_{22}}{\rho_{11}} = e^{-\hbar\omega_0/k_B T}$. Einstein thinks the excited state population is maintained by blackbody radiation; Max Planck derived the spectrum of blackbody radiation as $I(\omega) = \frac{\hbar\omega^3}{4\pi^3 c^2}\frac{1}{e^{\hbar\omega/kT}-1}$ just few years ago.

A. Assume the blackbody radiation is weak $kT << \hbar\omega_0$ and atomic decay rate is slow $\Gamma << \omega$, we can estimate the excited state population due to blackbody radiation as $\rho_{22} = \frac{P}{2} = \frac{1}{2}\int \frac{I(\omega)/I_S}{1+4(\omega-\omega_0)^2/\Gamma^2} d\omega$. Show that this amounts to $I_s = \frac{\hbar\omega^3\Gamma}{16\pi^2 c^2}$.

B. Another consequence of Einstein's idea is about the radiative cross section of an atom. Consider that the scattering rate is $s = \Gamma\rho_{22} = \sigma\frac{I}{\hbar\omega}$. Show that the maximum cross section for an atom is solely given by the wavelength of the resonance transition λ as $\sigma_{max} = 2\lambda^2$.

(Remark: A full quantum mechanical calculation gives $I_s = \frac{\hbar\omega^3\Gamma}{12\pi c^2}$ and $\sigma_{max} = \frac{3}{2\pi}\lambda^2$.)

2. Zeeman slower
A widely used scheme to slow down room temperature atoms is pointing a laser beam directly against their motion. As an atom in the beam scatters photons, they slow down by the photon momentum. A Zeeman slower is developed to slow down atoms from an oven at high temperature toward zero so they can be captured by the magneto-optical trap. The Zeeman slower is sketched below



Here the atoms are ejected from the oven with a typical initial velocity of 300m/s, slowed by the Zeeman slower beam, and captured in a MOT. The Zeeman slower magnetic field is used to cancel the Doppler shift for atoms moving at high velocity.

A. Assuming the Zeeman beam is tuned to the resonance frequency of a stationary Rubidium atom $\omega = \omega_0$, show that the laser can only excite atoms with tiny velocity $|v|$ < 4 m/s.

(Hint: Doppler shift is $\Delta\omega = \omega\frac{v}{c}$. For Rubidium atoms, the excitation state linewidth is $\Gamma = 2\pi \times 5\text{MHz}$, resonance wavelength is $\lambda = 2\pi c / \omega_0 = 780\,\text{nm}$)

B. In order to continuously slow the atoms from v_i=300 m/s to 0 m/s, the Zeeman slower is designed by introducing a weird shaped magnetic coil to cancel the Doppler shift throughout the slowing process. The magnetic field introduces a Zeeman shift $\mu_B B = \hbar\Delta\omega$, where μ_B is the magnetic moment of the transition. Show that if the Doppler shift is perfectly cancelled, one can decelerate atoms with the maximum acceleration of $a = -\frac{\Gamma\hbar\omega}{2mc}$, where m is the atomic mass. What is the condition to reach the maximum deceleration in a slower?

C. A Zeeman slower is engineered to reach the maximum slowing efficiency in the shortest length. Given the initial velocity of the Rubidium atom v_i, show that the magnetic field distribution along the Zeeman slower is given by $B(z) = B_0\sqrt{1-\frac{z}{L}}$. Determine the maximum field needed B_0 and the length of the slower L.

3. Dressed atom picture

In the class, we discuss $|g, n+1>$ can couple to $|e, n>$, where *n=0,1,2…* is the number of photons in the cavity and *g/e* refers to atom in the ground/excited state. The dressed atom Hamiltonian under RWA is

$$H = \hbar\omega_0 a^+ a + g(a^+ b + a b^+) + (\hbar\omega + \frac{1}{2})b^+ b,$$

where a^+ is the creation operator of atomic excitation and b^+ is the creation operator of a photon.

A. First we check whether this Hamiltonian is consistent with the semi-classical Hamiltonian when the photon number is large $<b^+ b> \equiv n >> 1$, namely, $H = -\frac{\Delta}{2}\sigma_z + \frac{\Omega}{2}\sigma_x$. Write down the dressed atom Hamiltonian in the basis of $|g,n>$ and $|e,n-1>$:

$$H = \begin{pmatrix} <g,n|H|g,n> & <g,n|H|e,n-1> \\ <e,n-1|H|g,n> & <e,n-1|H|g,n-1> \end{pmatrix}.$$

Show that the two Hamiltonians are the same as long as $g = \frac{\Omega}{2\sqrt{n}}$. What is the physical meaning of g?

A. The energy structure would be quite different for small photon number *n*. Draw the energy levels of the lowest eigenstates $|g, n=0>$, $|e,0>$, $|g,1>$, $|e,1>$, $|g,2>$, $|e,2>$ with zero and finite coupling strength g. (You may assume Δ>0).

B. Now let's include spontaneous emission, which permits decays from $|e,n>$ to $|g,n>$ in the bare state basis. Draw wiggling lines to show how an atom can decay toward lower states. How many distinct frequencies of the emitted photons can you see if the atoms are initially prepared in $|e,2>$?

C. Now we can answer an important question, namely, how can an atom be found in the excited state with finite probability when it is illuminated by photons with insufficient energy to excite, namely, $\hbar\omega < \hbar\omega_0$? Wouldn't such observation violate energy conservation?
Consider the complete atom+photon system. We place an atom in the eigenstate *|e,n+1>*, what is the frequency of the spontaneously emitted photon when the atom decays back to ground state? Argue how you can save the energy conservation based on the dressed atom picture?