P238 Atomic Physics 2018 Cheng Chin

Home Work 4

(Due: 11/27/2018)

Bose-Einstein condensation

Bosons experience Bose-Einstein condensation (BEC) when there are more than on average 1 or more particles in a given quantum state. This condition requires very low temperatures below << 1 micro-Kelvin in typical cold atom experiments. Here we will figure out the condition to reach BEC.

Consider *N* particles at temperature *T* in a large 3-dimensional box with volume $V=L^3$, we may estimate the population in a quantum state as $N_i = \frac{N}{Z} e^{-\beta E_i}$ (Boltzmann distribution), where $\beta^{-1} = k_B T$, E_i is the energy of the $i-$th quantum state, and the single-particle partition function $Z = \sum_i e^{-\beta E_i}$ is derived from the normalization condition $N = \sum_i N_i$.

A. Argue that the population in the ground state $N_0 = \frac{N}{Z} e^{-\beta E_0}$ is always higher than those in any other states, and thus when the temperature reduces, BEC will always occur in the ground state population.

B. To calculate N_0 , we need to compute the partition function $Z = \sum_i e^{-\beta E_i} = \int_0 e^{-\beta E} n(E) dE$, where we have transformed the sum into an integral and $n(E)dE$ is the number of quantum states with energies between E and $E + dE$. For a large box, derive the density of state $n(E) = \frac{4\pi V \sqrt{2m^3 E}}{h^3}$.

(Hint: in a box, particle momentum $P = \hbar(k_x, k_y, k_z) = \frac{h}{2L}(n_x, n_y, n_z)$ is quantized with $n_x, n_y, n_z = 1, 2 \ldots$ and $E = \frac{p^2}{2m}$. You can omit zero-point energy in the box.)

C. Now we can perform the integral. Show that the condition to reach $N_0 \geq 1$ reduces to simply $n_\phi \equiv n\lambda_{dB}^3 \geq 1$, where $n = \frac{N}{V}$ is the particle density, $\lambda_{dB} = \frac{h}{\sqrt{2\pi m k_B T}}$ is the thermal de Broglie wavelength and n_ϕ is called the phase space density. Thus the critical temperature is $T_c = \frac{n^{\frac{2}{3}} h^2}{2\pi m k_B}$

D. The above result looks very different from what we obtained in the class, namely, in a harmonic trap $V(r) = \frac{1}{2} m\omega^2 r^2$, we showed that the onset of BEC is $T = T_c = N^{\frac{1}{3}} \hbar\omega / k_B$, where ω is the trap frequency. Show that the two conditions are actually exactly identical if you evaluate the atomic density at the center of the harmonic trap.

(Hint: Evaluate the peak density of an ideal gas with *N* atoms in the harmonic trap at temperature $T_c = N^{\frac{1}{3}}\hbar\omega/k_B$, and show that the density satisfies the result from C.)

E. Argue that the consistency of the results of C and D suggests that <u>BEC is quantum phenomena that depends only on the local thermodynamic quantities</u>. Namely, when one releases a collection of *N* bosons in a generic potential *V(x),* and a density distribution of the particles *n(x)* is reached in thermal equilibrium at temperature *T*, BEC should likely occur whenever and wherever the condition $n(x)\lambda_{dB}^3 \geq 1$ is reached. Is such conjecture true or can you find a counter-example?
(Hint: Can you find a potential *V(x)* such that you know BEC would not occur even when the local density satisfies $n(x)\lambda_{dB}^3 \geq 1$?)