1. **Dirac Delta function** $\delta(x)$ (11 points each)
   We may define the Delta function based on the following procedure
   
   • $f(x)$ is any function that has an integrated area of $\int f(x)dx = 1$.
   • Dirac’s delta function is defined as $\delta(x) = \lim_{\Delta \to 0} \frac{1}{\Delta} f(\frac{x}{\Delta})$

   (a) A common choice of $f$ by physicists is the Gaussian function $f(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$. Use the form and prove that $\delta(x)$ satisfies the following properties
   
   1. $\delta(x \neq 0) = 0$
   2. $\delta(x = 0)$ diverges
   3. $\int \delta(x)dx = 1$
   4. $\int g(x)\delta(x-a)dx = g(a)$.
   (Hint: Apply the definition.)

   (b) Calculate or prove the following
   
   1. $\int g(x)\delta(ax + b)dx$
   2. $\int g(x)\frac{d\delta(x)}{dx}dx$, $g(\pm\infty) = 0$
   3. $\delta(x - a) + \delta(x + a) = 2|a|\delta(x^2 - a^2)$.

   (c) Solve the motion of a simple harmonic oscillator which is periodically kicked at period 1. Determine the condition that the harmonic oscillator becomes unstable.

   $$x'' + \omega_0^2 x = f(t) \equiv \sum_{n=-\infty}^{\infty} \delta(t - n)$$

   (Hint: the standard approach is to start with Fourier expanding the external force. You may also use the fact that the integral $\int_{n-\epsilon}^{n+\epsilon} \delta(t - n)dt = 1$ for arbitrary small interval of $2\epsilon$.)

2. **Energy and energy flow in a wave** (11 points each)
   Here we will investigate how is energy distributed and propagating in a generic medium (string, air, water...). Assume the wave (transverse or longitudinal) satisfies the following wave equation

   $$\rho \frac{\partial^2 \psi(x,t)}{\partial t^2} = T \frac{\partial^2 \psi(x,t)}{\partial x^2},$$

   where $\rho$ is the linear density of the medium and $T$ is the “tension force” in the medium.

   (a) Given a small section between $x$ and $x + \Delta x$, calculate its kinetic energy density $\partial_x E_k$ and potential energy density $\partial_x U$. 
(Hint: First determine what the “potential energy” is for the section. Think about how the restoring force $-kx$ is linked to the potential energy $\frac{kx^2}{2}$ in a SHO.)

(b) Given a traveling wave $\psi(x, t) = A \cos k(x - vt)$, what is the total energy density $\partial_x E = \partial_x E_k + \partial_x U$. Explain why the total energy is time-dependent and is zero when the magnitude of the displacement $|\psi|$ reaches the maximum value, where we expect to see large potential energy?

(c) Given the traveling wave $\psi(x, t) = A \cos k(x - vt)$, determine the amount of energy that passes the position $x$ per unit time in the propagation direction. This energy transfer (energy flux) can be written as $J = T \partial_x \psi \cdot \partial_t \psi$. Show that the above results suggest

$$J = v_E \partial_x E,$$

where the velocity of the energy flow is the same as that of the waveform $v_E = v = \frac{\sqrt{T}}{\rho}$.

3. Decibel scale of the strength of sound (11 points each)

Alexander Bell, the inventor of phone, introduced the unit of bel, which became decibel in acoustics: Zero decibel (0 dB) is defined as $\pm 20 \mu Pa$ in the variation of air pressure ($\mu Pa = 10^{-6} Pa$), the typical threshold of human perception. Decibel is presented in log scale such that 20 dB corresponds to 200 $\mu Pa$, 40 dB 2 $m Pa$, 60 dB 20 $m Pa$ and so on.

(a) What is the assumed threshold of human perception? Calculate the intensity of a 1D acoustic wave at 0 dB.

(Hint: Intensity is energy delivered per area per time in the unit of Watt/m$^2$.)

(b) Our hearing is damaged above 100 dB, calculate the power of the wave. Show that sound cannot be louder than 200 dB.

(c) How much is the maximum displacement $\psi(x)$ of air molecules away from equilibrium in the presence of acoustic waves at 0 dB and frequency = 100 Hz?

<table>
<thead>
<tr>
<th>Material</th>
<th>Density (kg/m$^3$)</th>
<th>Compressibility (1/GPa)</th>
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<tbody>
<tr>
<td>Air</td>
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<tr>
<td>Water</td>
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<tr>
<td>Copper</td>
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<td>0.0073</td>
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</tbody>
</table>

(GPa = 10$^9$ Pa.)