1. (Math) Vector calculus (20 points each)
   a) A vector field \( \vec{A}(x, y, z) \) carries a source or sink if \( \nabla \cdot \vec{A} \neq 0 \) and it carries circulation if \( \nabla \times \vec{A} \neq 0 \). Determine which function(s) carry no sources, no sinks and no circulation?
      \( \vec{A} = (e^x, 1, x e^y) \)
      \( \vec{B} = (x^2 y, -xy^2, 0) \)
      \( \vec{C} = (yz, zx, xy) \)
      \( \vec{D} = (\cos z, \sin x - \sin z, -\cos x) \)

   b) Prove the following vector identities (\( \phi \) is a scalar field, \( \vec{A} \) is a vector field.)
      \( \nabla \cdot (\phi \vec{A}) = \phi \nabla \cdot \vec{A} + (\vec{A} \cdot \nabla) \phi \)
      \( \nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} \)
      \( \nabla \cdot (\phi \nabla \psi - \psi \nabla \phi) = \phi \nabla^2 \psi - \psi \nabla^2 \phi \)

2. Doppler effect and shock waves (20 points)
   Police uses Doppler radar gun to identify speeding violators. Assume the gun emits sound waves at frequency \( \omega \), and police detects the frequency of the reflected waves as \( \omega_r \).
   a) A car is moving straight away from the police at the speed \( v \), show that the frequency of the reflected radio waves has a frequency of
      \[ \omega_r = \omega_0 \frac{v_p - v}{v_p + v} \]
   b) In the presence of strong wind, the above formula needs corrections. Assume the wind is in the same direction as the car and its speed is \( w \), show that the reflected frequency is
      \[ \omega_r = \omega_0 \frac{v_p + w}{v_p} \frac{v + w}{v_p - v + w} \]
   c) Mach number is the ratio of the vehicle speed to the sound speed \( M = \frac{v}{v_p} \). For a supersonic jet, show that the Mach number can be determined from the shock angle \( \theta \) as \( \sin \theta = \frac{1}{M} \). How fast is the jet moving in the photo with \( \theta = 58^\circ \) and sound speed = 343 m/s?