Physics 143b: Honors Waves, Optics, and Thermo
Spring Quarter 2020
Problem Set #8
Due: 11:59 pm, Thursday, June 4. Please submit to Canvas.

For more reading material, please see Introduction to Thermal Physics by Schroeder. (You can find pdf online.)

1. Ideal gas (8 points each)
   The laws we learned about the ideal gas are $PV = NkT$ and internal energy of the gas $U = \gamma NkT$, where $\gamma = 1.5$ for atomic gas and 2.5 for molecular gas. We may investigate other properties about the ideal gas.
   a) Heat capacity:
      Given the isochoric specific heat $c_V = \frac{1}{M} \frac{\partial Q}{\partial T} \bigg|_V$ we derived in Lecture 15-1, show that we can write the internal energy of the gas as
      \[ U = c_V N m T, \]
      where $m$ is the atomic/molecular mass.
   b) Compressibility:
      In the derivation of sound speed, we introduce the compressibility as the fractional reduction of the volume per unit pressure applied to the system:
      \[ \beta = -\frac{1}{V} \frac{\partial V}{\partial P}, \]
      We are now in good position to derive it. Show that the isothermal and isentropic compressibility of an ideal gas are $\beta_T = -\frac{1}{V} \frac{\partial V}{\partial P}_T = \frac{1}{P}$ and $\beta_S = -\frac{1}{V} \frac{\partial V}{\partial P}_S = \frac{\gamma}{(\gamma+1)P}$.
      Here isothermal means the temperature is kept constant, while isentropic means the process is adiabatic and reversible, a condition derived in Lecture 15-1. The sound speed we studied is based on the isentropic compressibility. Give your argument why $\beta_S$ gives a better prediction on the sound speed than $\beta_T$?
   c) Thermal expansion:
      Thermal expansion determines the fractional change of the system when the temperature increases by 1 unit. There are again two possible processes: isobaric thermal expansion coefficient $\alpha_V = \frac{1}{V} \frac{\partial V}{\partial T}_p$ and isochoric coefficient $\alpha_p = \frac{1}{P} \frac{\partial P}{\partial T}_V$. Show that both lead to the same result as $\alpha_V = \alpha_p = \frac{1}{T}$.
      How much does the mean molecular distance in an ideal gas increases fractionally when temperature increases from 300K to 301K?
2. Entropy and adiabaticity (10 points each)

We showed in Lectures 15-1 and 15-2 that when an ideal gas slowly evolves adiabatically, it follows \( PV^{(\gamma+1)/\gamma} = \text{const}. \) Here we will show that the constant is linked to the entropy of the system and the process is isentropic (entropy main a constant).

a) Use \( dU = TdS - PdV \) and show that the entropy change from state 1 to state 2 is given by

\[
S_2 - S_1 = c_v N k \ln \frac{P_2 V_2^{(\gamma+1)/\gamma}}{P_1 V_1^{(\gamma+1)/\gamma}}.
\]

Hint: One can recast the equation as a differential equation \( dS = \frac{dU}{T} - \frac{P dV}{T} \). Then use the ideal gas law to integrate the differential equation.

To get the explicit form of entropy, we note that in the zero temperature limit, we have \( T = 0 \) (3rd law of thermodynamics) and so are the pressure, volume and internal energy of an idea gas be zero. Derive the form of entropy with the above condition and show that it is compatible with the Sackur-Tetrode equation

\[
S = N k \ln \left( \frac{5 V}{2 N (3 \hbar^2 / N)^{3/2}} \right).
\]

Hint: The explicit form of the entropy requires Planck constant \( h \). One may only imagine Mr. Dr. Sackur could have figured out the value of \( h \) decades before Max Planck.

b) Show that in an adiabatic process \( PV^{(\gamma+1)/\gamma} = \text{const}. \) the entropy of the system conserves. Determine the explicit form of the constant in terms of entropy.

3. Maxwell-Boltzmann distribution (10 points each)

A great insight from Maxwell is that the velocity distribution of an ideal gas follows \( P(v) = p(v_x)p(v_y)p(v_z) \), where \( p(v) \propto e^{-E_k/kT} \) is the Maxwell-Boltzmann distribution.

a) Given the probability conservation condition \( \int_{-\infty}^{\infty} p(x) dx = 1 \) for a stochastic variable \( x \), determine the explicit form of \( p(v_x) \).

b) Show that the probability distribution of \( P(v) \), where \( v = \sqrt{v_x^2 + v_y^2 + v_z^2} \) is the absolute value of the molecular velocity, is given by

\[
P(v) = \left( \frac{m}{2 \pi k T} \right)^{3/2} 4 \pi v^2 e^{-mv^2/2kT},
\]

which is the most common form of Maxwell-Boltzmann distribution.
Hint: A coordinate transform from Cartesian coordinate $(x, y, z)$ to spherical coordinate should satisfy probability conservation. Assume the probability to find a particle in space within a small volume element $dv$ is $dP = P(\vec{r})DV$, the same probability should be found in a new coordinate system, namely,

$$dP = P(\vec{r})dx dy dz = P(\vec{r})r^2 \sin \theta d\theta d\phi \equiv P(r, \theta, \phi)dr d\theta d\phi.$$  

c) Determine the root-mean-square velocity $v_{rms} = \sqrt{<v^2>}$ and mean velocity $<v>$ of an ideal gas. Show that $v_{rms} = \sqrt{\frac{3kT}{m}}$ is consistent with the equipartition theorem. The mean velocity $<v>$ is, however, smaller. Compare the velocity to the sound speed, can air molecules move faster than sound and thus be supersonic?

4. Refrigeration (10 points each)

The refrigerator is an device that receives energy (electrical energy for standard refrigerators) from the “Source of work” in order to extract heat from the source at a low temperature $T_c$ and deliver the energy into the heat sink at $T_H$ (typically the atmosphere).

![Refrigeration Diagram]

a) Assume the above process is based on Carnot cycle running in reverse. Determine how much energy $W$ is demanded in order to extract one Joule of energy from the cold source.

Hint: Calculate $\frac{dW}{dQ_c}$ according to the definition in the diagram. You may just use the results we derived in Lecture 15-2 and assume the processes are now running backwards. Show the result in terms of the temperatures $T_H$ and $T_C$.

b) Your fridge keeps the food at around 40 F, while the ambient temperature is 72 F. How many Joules are needed to remove 1 Joule of energy from the stuffs in the fridge?

c) How far can we cool? One can now cool atoms to nano-Kelvins. Show that if we treat the atom as the heat source at $T_C$ and the lab as the heat sink, the amount of energy required to cool an atom from $T_C = T_H = 300 \ K$ to $T_C = 10^{-9}$ using the reverse Carnot cycle would be

$$W = c_v k \left[ T_H \left( 1 - \ln \frac{T_H}{T_C} \right) - T_C \right]$$

d) Show that energy required to cool an atom toward zero temperature $T_c \to 0$ diverges logarithmically as $W_{cool} = c_v k T_H \ln \frac{T_H}{T_C}$.

Hint: This result is another demonstration that zero temperature is not attainable.