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About the class:
- Syllabus: ultrawld.uchicago.edu
- Asynchronous teaching: videos
- Labs: Phys. Dept. Staffs
- Synchronous teaching: Zoom (143)
- Discussion and office hours: piazza
- Grades distribution

Homeworks:
- Released on website.
- Due in one week.
- Submit HWs to Canvas. (Also lab reports & Midterm)
- Late policy: -10% per day.

Midterm: 24 hours take home. Submit to TAs.

Final: not yet decided

Questions?
- Class, Lectures, discussion → Email Cheng
- HWs, Midterm and final → TAs
- Lab and reports → Lab Staffs and TAs.
Lecture 1-2 Solving ordinary differential equations

The eqn that concerns us many times in this class will be of the following form:

\[ A x''(t) + B x'(t) + C x(t) = D f(t), \quad A, B, C, D \text{ are constants} \]

also written as

\[ A \frac{d^2x(t)}{dt^2} + B \frac{dx(t)}{dt} + C x(t) = D f(t) \]

This is called \textit{2nd order ordinary differential equation (ODE)} only up to 2nd derivatives one independent variable \( t \)

Example 1: \( F = ma = m \ddot{x}(t) \Rightarrow \ddot{x}(t) = \frac{F}{m} \) is a 2nd order ODE

Example 2: Point mass \( m \) attached to a spring with restoring force

\[ F = -kx \]

\[ \Rightarrow \dot{x}(t) = -\frac{k}{m} x(t) \]

\[ \Rightarrow \dot{x}' + \frac{k}{m} x = 0 \]

This is a \textit{homogeneous 2nd order ODE} \( \text{no } f(t) \text{ on the right side} \)

Last definition: A differential eqn is called linear if ....

\( x_1(t) \) is a solution, \( x_2(t) \) is also a solution

\[ \Rightarrow x = \alpha x_1(t) + \beta x_2(t) \text{ is also a solution} \]

We will deal with linear, homogeneous ODE in this class.

\[ A x''(t) + B x'(t) + C x(t) = 0 \]
\[ A \alpha x''_1(t) + B \alpha x'_1(t) + C \alpha x_1(t) = 0 \]
\[ A \beta x''_2(t) + B \beta x'_2(t) + C \beta x_2(t) = 0 \]
\[ X = \alpha x_1(t) + \beta x_2(t) \]

\[ \text{Show that } A x''(t) + B x'(t) + C x(t) = 0 \]
\[ \text{You can see this by substitution!!} \]
First case: Simple Harmonic Oscillator

\[ U = \frac{1}{2} k x^2 \]

\[ \Rightarrow F = -dU/dx = -kx \]

\[ \Rightarrow m x''(t) = F = -kx \]

\[ \Rightarrow x''(t) + \frac{k}{m} x(t) = 0. \]

\[ \Rightarrow x(t) = ? \]

A good tip: Try \( \sin \) and \( \cos \) (since we know the motion is periodic)

Plug in \( x = A \cos (\omega t + \phi) \)

\[ \Rightarrow x'' = -A \omega^2 \cos (\omega t + \phi) \]

\[ \Rightarrow \omega = \sqrt{\frac{k}{m}}, \quad A \text{ and } \phi \text{ are not constrained} \]

Similarly, \( A \sin (\omega t + \phi) \) also works

\[ A: \text{amplitude} \]

\[ \omega: \text{angular freq} \Rightarrow T = \frac{2\pi}{\omega}: \text{period}, \quad f = \frac{\omega}{2\pi}: \text{frequency (Hertz)} \]

Since eqn is linear we know \( A \cos \omega t + B \sin \omega t \) is also a solution. The values of \( A \) and \( B \) can be determined by the initial condition.

If initial position \( x(0) = 0 \), velocity \( x'(0) = u_0 \)

\[ \Rightarrow x(t) = A \sin \omega t, \quad A \omega = u_0 \]

If \( x(0) = x_0, \quad x'(0) = 0 \)

\[ \Rightarrow x(t) = x_0 \cos \omega t \]
How about the general case $Ax^2 + Bx + C = 0$?

A great tip: Try $x(t) = e^{at}$

$$x(t) = ae^{at}$$

$$x''(t) = a^2 e^{at}$$

(Why? b/c exp function looks the same after differentiation.)

Plugin, we get $Aa^2 + Be^{at} + C = 0$.

Since $e^{at} \neq 0$, we get $Aa^2 + Ba + C = 0$.

We reduce ODE into an algebraic Eqn!!

We have 2 roots: $a = \frac{1}{2A} \left[ -B \pm \sqrt{B^2 - 4AC} \right] = a_1, a_2$

So the general solution is $x(t) = e^{a_1 t} + e^{a_2 t}$.

Again, $a_1$ and $a_2$ can be determined by the initial conditions.

Let's try this tip on harmonic oscillator: $x''(t) + \frac{k}{m} x(t) = 0$

Given $x(t) = e^{at}$, we get $a^2 e^{at} + \frac{k}{m} e^{at} = 0$.

$$a^2 = -\frac{k}{m} \Rightarrow a = \pm \frac{\sqrt{k}}{\sqrt{m}} i = \pm \omega i$$

$$c = -1$$

$$\Rightarrow \text{General solution is } x(t) = A e^{i \omega t} + B e^{-i \omega t}$$

$A$ and $B$ are arbitrary constants.

This is identical to $x(t) = A \cos \omega t + B \sin \omega t$, $A$ and $B$ are constants, since

$$e^{i \omega t} = \cos \omega t + i \sin \omega t$$

$$e^{-i \omega t} = \cos \omega t - i \sin \omega t$$

equivalently

$$\cos \omega t = \frac{e^{i \omega t} + e^{-i \omega t}}{2}$$

$$\sin \omega t = \frac{e^{i \omega t} - e^{-i \omega t}}{2i}$$
Notes on complex numbers $z = a + bi$, $i^2 = -1$, $a, b$ are real.

$a = \text{Re}[z]$ is the real part

$b = \text{Im}[z]$ is the imaginary part.

**Exponential representation**

$z = r e^{i \theta} = r \cos \theta + i r \sin \theta$

$r = |z|$ is the absolute value

$\theta = \text{Arg}(z)$ is the argument

**Graphic representation**

\[ i = e^{i \pi/2} \]

\[ z = r e^{i \theta} = r (\cos \theta + i \sin \theta) \]

Very cool consequences:

1. $e^{i \theta_1} \times e^{i \theta_2} = ?$

   **$z_1 \times z_2$**

   Stupid method:

   \[
   (\cos \theta_1 + i \sin \theta_1) \times (\cos \theta_2 + i \sin \theta_2)
   = \cos \theta_1 \cos \theta_2 - i \sin \theta_1 \sin \theta_2 + i \cos \theta_1 \sin \theta_2 + i \sin \theta_1 \sin \theta_2
   \]

   \[
   = \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 + i (\cos \theta_1 \sin \theta_2 + \sin \theta_1 \cos \theta_2)
   \]

   **Slow**

   Smart method: $e^{i \theta_1 + i \theta_2} = \cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2)$

2. $\frac{z_2}{z_1} = \frac{r_2 e^{i \theta_2}}{r_1 e^{i \theta_1}} = \frac{r_2}{r_1} e^{i \theta_2 - i \theta_1} = \frac{r_2}{r_1} e^{i (\theta_2 - \theta_1)}$

3. $1 = e^0$, $i = e^{i \pi/2}$, $-1 = e^{i \pi}$, $-i = e^{-i \pi/2}$
Lecture 1-3  Simple Harmonic Oscillator  Cheng Chin

What are waves?

- Water waves, sound waves, light waves, seismic waves, gravitational waves, shock waves, brain waves, matter waves, hand waves

Waves are modulation of physical quantities in space & time.

It occurs frequently when the system is driven slightly away from "equilibrium".

Simple harmonic oscillator (SHO) model

We can expand a generic potential energy around its local minimum by Taylor expansion

\[ V(x) = V(x_0) + V'(x_0)(x-x_0) + \frac{1}{2} V''(x_0)(x-x_0)^2 + \frac{1}{6} V'''(x_0)(x-x_0)^3 + \ldots \]

Constant = 0 since slope = 0 leading order higher order terms

If we only perturb the system gently, we ignore higher orders & get...
\[ V(x) = V(x_0) + \frac{1}{2} V''(x_0) (x-x_0)^2. \] If we further shift the origin to the equilibrium position \((x_0, V(x_0))\), we get

**Harmonic Potential**

\[ V(x) = \frac{1}{2} k x^2, \quad k = V''(x_0) \]

Small deviation from equilibrium

Restoring force \( F = -dV/dx = -kx \) (Hooke's law)

Motion of particle near the bottom:

\[ F = ma = m x''(t) = -kx(t) \]

\[ x(t) = A \cos \omega t + B \sin \omega t, \quad \omega = \sqrt{k/m} \]

or equivalently \( x(t) = \bar{A} \cos (\omega t + \phi) \), or \( x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t} \)

Note: There are 2 independent constants determined by the initial conditions. Two because this is a 2nd order differential Eqn.

**Diagram:**
- **Amplitude** \( \bar{A} \)
- **Period** \( T = \frac{1}{f} = \frac{2\pi}{\omega} \)
- **Frequency** \( f \) in Hz
- **Angular Frequency** \( \omega \) in 1/s

\[ \text{time } t \]
Given two initial conditions, we can determine $A$ and $B$
($\overline{A}$ and $\phi$, $c$, and $c_2$)

$$x = A \cos \omega t + B \sin \omega t$$

Say we know initial position $x(0) = x_0$, velocity $\dot{x}(0) = v_0$

$$\Rightarrow x(0) = A \cos 0 + B \sin 0 = A = x_0$$
$$\dot{x}(t) = x'(t) = -A \omega \sin \omega t + B \omega \cos \omega t$$
$$\dot{x}(0) = -A \omega \sin 0 + B \omega \cos 0 = B \omega = v_0$$

$$\Rightarrow A = x_0, B = \frac{v_0}{\omega}$$

* Here $x(t) = x_0 \cos \omega t + \frac{v_0}{\omega} \sin \omega t$ is a particularly simple form when we know $x_0 \leq v_0$.

If we know the amplitude and phase, you would prefer $x(t) = A \cos (\omega t + \phi)$. In other cases, the complex form is better.

Phase diagram $x(t) = A \cos (\omega t + \phi) = \text{Re} \left[ A e^{i(\omega t + \phi)} \right]$

$= x$-axis projection of $A e^{i(\omega t + \phi)}$

rotating as time evolves $\dot{\theta} = \frac{\pi}{2}$

All rotating in sync!!!
We can see when $x$ reaches maximum $x=A$, velocity is 0. When velocity reaches maximum, displacement is zero! (Same applies to velocity and acceleration.)

**Energy of Simple harmonic oscillator:**

- Potential energy: $V = \frac{1}{2} kx^2 = \frac{1}{2} kA^2 \cos^2(\omega t + \phi)$
- Kinetic energy: $E_k = \frac{1}{2} mv^2 = \frac{1}{2} m\omega^2 A^2 \sin^2(\omega t + \phi)$

Note that $\omega = \sqrt{\frac{k}{m}}$, so we have total energy:

$$E = V + E_k = \frac{1}{2} kA^2 \cos^2(\omega t + \phi) + \frac{1}{2} kA^2 \sin^2(\omega t + \phi)$$

$$= \frac{1}{2} kA^2 = \text{const. of time}$$

( Energy conservation )

**Example:** In our lab cesium atoms are trapped in a laser beam the generates a Gaussian potential.

\[ V = -V_0 e^{-2r^2/r_0^2} \]  

**Calculate the vibration period near equilibrium.**

\[ F = -\frac{dV}{dr} = -V_0 2r/r_0^2 e^{-2r^2/r_0^2} \approx -\frac{2V_0 r}{r_0^2} \]

\[ m \ddot{r} + \frac{2V_0}{r_0^2} r = 0 \]

\[ \omega = \sqrt{\frac{2V_0}{m r_0^2}} = \frac{1}{r_0} \sqrt{\frac{2V_0}{m}} \]

\[ \Rightarrow \text{period} = \frac{2\pi}{\omega} = 2\pi r_0 \sqrt{m/V_0} \Rightarrow \text{Short period in small beam.} \]
First a demonstration that most oscillators damp out because of dissipation.

\[ F = -b \dot{u} = -b x'(t) \]

\[ m x'' = -k x - b x' \Rightarrow x'' + \gamma x' + \omega_0^2 x = 0 \]

We define \( \gamma = b/m \): friction coeffi.
\( \omega_0 = \sqrt{k/m} \): natural freq.

\[ \Rightarrow \text{So this lecture we will discuss} \]
\[ x''(t) + \gamma x'(t) + \omega_0^2 x(t) = 0 \]

Use the exp in Lecture 1-2, \( x(t) = e^{\alpha t} \), \( x'(t) = \alpha e^{\alpha t} \), \( x'' = \alpha^2 e^{\alpha t} \)

\[ \Rightarrow \alpha^2 + \gamma \alpha + \omega_0^2 = 0 \quad \text{quadratic eqn.} \]

\[ \Rightarrow \alpha = \frac{-\gamma \pm \sqrt{\gamma^2 - 4 \omega_0^2}}{2} = \alpha_1, \alpha_2 \]

\[ \Rightarrow x(t) = A e^{\alpha_1 t} + B e^{\alpha_2 t}, \text{ where } A \text{ and } B \text{ can be} \]

\[ \text{determined by } 2 \text{ initial conditions.} \]

So what's new here?

- \( \alpha_1, \alpha_2 \) can be complex \( \gamma < 4 \omega_0^2 \) (underdamping)
- \( \alpha_1, \alpha_2 \) can be real \( \gamma > 4 \omega_0^2 \) (overdamping)
- \( \alpha_1 = \alpha_2 \) can be identical \( \gamma = 4 \omega_0^2 \) (critical damping)
- \( \alpha_1 \text{ and } \alpha_2 \) can be purely imaginary \( \gamma = 0 \Rightarrow \text{SHO} \)

We will discuss the first 3 cases here:

\* \( \gamma > 0, \omega_0 > 0 \), we do not consider unphysical case \( \gamma < 0 \)
Underdamped oscillator (\(\gamma < 2\omega_0\)) \(\Rightarrow \sqrt{\gamma^2 - 4\omega_0^2} = \frac{\sqrt{4\omega_0^2 - \gamma^2}}{2}\)

\[
\alpha = \frac{1}{2} \left( -\gamma \pm \sqrt{4\omega_0^2 - \gamma^2} \right) = \frac{-\gamma}{2} \pm \frac{i}{2} \sqrt{4\omega_0^2 - \gamma^2} = -\frac{\gamma}{2} \pm i\mu = \alpha_1, \alpha_2
\]

\[
x(t) = C_1 e^{-\frac{\gamma t}{2}} \cos(\mu t) + C_2 e^{-\frac{\gamma t}{2}} \sin(\mu t) = e^{-\frac{\gamma t}{2}} (C_1 \cos(\mu t) + C_2 \sin(\mu t))
\]

equivalently

\[
e^{-\frac{\gamma t}{2}} (A \cos(\mu t + \phi))
\]

\[
= \Re \left( e^{-\frac{\gamma t}{2}} e^{i(\mu t + \phi)} \right)
\]

This part is the same as we saw in SDO.

All of them have 2 degrees of freedom.

Choose the form that is easiest for your calculation.

How does it look?

Amplitude \(A e^{-\frac{\gamma t}{2}}\) drops exponentially.

Period = \(\frac{2\pi}{\mu}\)

Angular freq. \(\mu = (\omega_0 - \frac{\gamma^2}{4})^{\frac{1}{2}} < \omega_0\)
Summary: Amplitude \( A e^{-\gamma t/2} \) decays at rate \( \gamma/2 \). Frequency shifts down to \( (\omega_0^2 - \gamma^2/4)^{1/2} \).

Total energy \( E = \frac{1}{2} k x^2 + \frac{1}{2} m x'^2(x) \)

Show that \( E(t) = E(0) e^{-\gamma t} \left[ 1 + \frac{\gamma^2}{4 \omega_0^2} \cos(2\pi t + \phi) \right] \)

Averaging over half cycle we get \( \langle E \rangle = E(0) e^{-\gamma t} \)

⇒ Energy damps out at the rate of \( \gamma \).

How many oscillations do we see before amplitude drops by \( e \)?

Define quality value a.k.a. Q factor \( Q = \frac{\omega_0}{\gamma} \) (dimensionless)

\[ A(t) = A(0) e^{-\gamma t/2} = A(0) e^{-\gamma t/2} \Rightarrow t = \frac{\gamma}{2} \]

We know period \( \omega = \frac{2\pi}{\omega_0} \approx 3\pi/2.5 \times 3.14 \)

⇒ So if you see 100 oscillations ⇒ \( Q \approx 300 \), \( \gamma = \frac{\omega_0}{300} \) good oscillator!!

if you see 1 oscillation ⇒ \( Q = 3.14 \) Barely an oscillator

Super-duper oscillators: Crystal oscillator \( Q = 10^4 \sim 10^6 \)

Super-conducting resonators: \( Q = 10^6 \sim 10^8 \)

Cs atomic clock: \( Q = 10^{10} \)

For \( Q = \frac{\omega_0}{\gamma} \gg 1 \), oscillation freq \( \omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} \approx \omega_0 \left( 1 - \frac{\gamma^2}{8 \omega_0^2} \right) = \omega_0 \left( 1 - \frac{1}{8Q^2} \right) \)

When \( Q = 100 \), frequency shifts by \( \frac{1}{8Q^2} = 10^{-5} = 0.001\% \)
Overdamped oscillator ($\gamma > 2\omega_0$) So $Q = \frac{\omega_0}{\gamma} < \frac{1}{2}$!!

Here the oscillations are so damped that it doesn't even oscillate:

$$\alpha = \frac{1}{2} \left[ -\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2} \right]$$

$\equiv \alpha_1$ and $\alpha_2$

both roots are real and negative

$$\alpha_1 = -\frac{\gamma}{2} \left[ 1 + \sqrt{1 - 4\omega_0^2/\gamma^2} \right] \equiv -M_1$$

$$\alpha_2 = -\frac{\gamma}{2} \left[ 1 - \sqrt{1 - 4\omega_0^2/\gamma^2} \right] \equiv -M_2$$

$$x(t) = Ae^{-M_1 t} + Be^{-M_2 t}$$

$M_1 > M_2 > 0$

• No more oscillation!!

• There are two ways to damp out the motion, one fast $e^{-M_1 t}$, one slow $e^{-M_2 t}$

Why are there 2 ways to damp out the motion?

Can we gain some intuition from the strong damping limit $\varepsilon = \frac{\omega_0^2}{\gamma^2} < 1$

$$M_1 = \frac{\gamma}{2} (1 + \sqrt{1 - 4\varepsilon}) \approx \gamma \text{ damping } = \gamma \text{ makes sense.}$$

$$M_2 = \frac{\gamma}{2} \left( 1 - \sqrt{1 - 4\varepsilon} \right) \approx \gamma \left[ 1 - (1 - 2\varepsilon) \right] = \varepsilon \gamma = \frac{\omega_0^2}{\gamma} \approx \frac{1}{\gamma}$$

With strong damping, there are 2 modes of motion:

One damps out fast when you increase damping.

One damps out slow when you increase damping.

If we wait for long time slow motion dominates.

Wierd!

What's going on??
Critical damping: \( \gamma = 2\omega_0 \)

\[ \alpha = \frac{1}{2} \left[ -\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2} \right] = -\frac{\gamma}{2} \],

double root!!

\[ x = Ae^{-\gamma t/2} = Ae^{-\omega_0 t} \]

problem!!

Well we have only one degree of freedom, how do we specify initial position \& velocity independently?

How do we handle \( x''(t) + 2\omega_0 x'(t) + \omega_0^2 x(t) = 0 \) ?

Math: when this happens, a special solution is \( x(t) = e^{\alpha t} \)

Prove that \( x(t) = te^{\omega_0 t} \) is indeed a solution

and \( \alpha = -\omega_0 \)

\( \Rightarrow \) general solution is \( x(t) = (A + Bt)e^{-\omega_0 t} \)

Physics: what's going on, how can a new solution suddenly appear? New type of dynamics?

What happens is that \( x(t) = Ae^{\alpha_1 t} + Be^{\alpha_2 t} \)

naturally approaches \( x(t) = A e^{\alpha_1 t} + B e^{\alpha_2 t} \) when \( \alpha_2 \to \alpha_1 \)

Pf:

\[ A e^{\alpha_1 t} + B e^{(\alpha_1 + \epsilon) t} \]

\[ = e^{\alpha_1 t} (A + B e^{\epsilon t}) \to e^{\alpha_1 t} (A + B(1 - e^t)) \]

\( \epsilon \to 0 \)

\[ = e^{\alpha_1 t} [A + B - Be^t] \]

\[ = (A + B e^{\alpha_1 t}) e^{\alpha_1 t} \]

\[ A = A + B, \quad B = -B e^{\alpha_1 t} \]
The solution is $x(t) = (A + Bt) e^{-\omega t}$.

* An interesting observation is that this solution passes zero once at $t = -A/B$.

(Assuming $B \neq 0$, it would not pass $x=0$ if $B=0$.)

$A e^{-\omega_0 t}$

Summary: Damped oscillator $x'' + \gamma x' + \omega_0^2 x = 0 \Rightarrow x = e^{-\omega t}$

$-\text{Re}[\alpha] > 0$: damping

$\text{Im}[\alpha] > 0$: frequency

$\omega_0 = \frac{\sqrt{\omega_0^2 - \gamma^2}}{2}$

$\frac{\gamma}{2} \left( 1 + \sqrt{1 - 4\omega_0^2} \right)$

$U \rightarrow \gamma$

damping $\rightarrow \gamma$

$\sqrt{\omega_0^2 - \gamma^2}$

overdamped $x = Ae^{-ut} + Be^{-u_2 t}$

underdamped $x = A e^{-\omega t/2} \cos(ut + \phi)$

critically damped $x = (A + Bt) e^{-\omega_0 t}$

Energy dissipates to zero the fastest.