Lecture 2-1 Solving Non-homogeneous ODE

Definitions
\[ x''(t) + \gamma x'(t) + \omega^2 x(t) = 0 \quad \text{homogeneous ODE} \]
\[ x''(t) + \gamma x'(t) + \omega^2 x(t) = f(t) \quad \text{non-homogeneous ODE} \]

1. One of the most important properties is that \( x_s + x_h \) is also a solution of the non-homogeneous ODE.

To prove it, just add the two equations and get
\[ (x_s + x_h)'' + \gamma (x_s + x_h)' + \omega^2 (x_s + x_h) = f(t) \]

Since we also know the homogeneous solution \( x_h = A e^{\alpha t} + B e^{\beta t} \)

\[ \Rightarrow \text{general form of the non-homogeneous ODE is} \quad x_s + x_h = x_s + A e^{\alpha t} + B e^{\beta t} \]

Since \( A \) and \( B \) take up degrees of freedom of \( x(0) \) and \( x'(0) \), the special solution \( x_s \) has no more degree of freedom. So it is special.

2. If \( x_{s1} \) is the special solution for \( f_1(t) \), \( x_{s2}(t) \) is the one for \( f_2(t) \), then \( x_{s1} + x_{s2} \) is the solution for \( f_1(t) + f_2(t) \).

This is because the qm is linear (prove it).

Say we want to solve \( x'' + \gamma x' + \omega^2 x = A \cos \omega t \) \((X, A \text{ real})\)

we can supplement it with \( y'' + \gamma y' + \omega^2 y = A \sin \omega t \) \((X, A \text{ real})\)

Add the two equations \( (x+iy)'' + \gamma (x+iy)' + \omega^2 (x+iy) = A \cos \omega t + i \sin \omega t \)

So we can instead solve \( \dot{X}'' + \gamma \dot{X}' + \omega^2 X = A e^{i\omega t} \)

and the real part of \( X \) is the solution \( X = \Re \{ X \} \), that we want.

Similarly, \( \Im \{ X \} \) is the solution with \( f(t) = A \sin \omega t \).
With complex \( \omega \), we can solve all equations at once.

But the real reason is that calculation is simple.

Driving force:

Let's solve \( x'' + \gamma x' + \omega^2 x = f \cos \omega t \) (driven damped harmonic oscillator) by solving \( \dddot{x} + \gamma \ddot{x} + \omega^2 x = \dot{f} \cos \omega t \) (angular \( \omega = \omega_0 \)).

Tip for special solution:

\[
\dddot{x} = A e^{i \omega t} \quad (X, A \text{ are complex})
\]
\[
\ddot{x} = A i \omega e^{i \omega t}
\]
\[
x' = -A \omega^2 e^{i \omega t}
\]

Plug in everything:

\[
-\omega \omega^2 e^{i \omega t} + A i \omega e^{i \omega t} + A \omega^2 e^{i \omega t} = \dot{f} e^{i \omega t}
\]

Since \( e^{i \omega t} \neq 0 \) \( \Rightarrow \)

\[
A \left[ -\omega^2 + i \omega \gamma + \omega^2 \right] = \ddot{f}
\]

\[
\Rightarrow x = \frac{f}{i \omega} e^{i \omega t}
\]

\[
\frac{1}{e^{i \phi}} = e^{-i \phi}
\]

The real part of \( \dot{x} \) is the desired solution.

\[
x_s = \text{Re} [\dot{x}] = \frac{f}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2}} \cos (\omega t - \phi), \text{ where}
\]

(Note that there is no more degrees of freedom, \( A \) and \( \phi \) are given)

Conclusion of the Math part:

Given \( x'' + \gamma x' + \omega^2 x = f \cos \omega t \) the solution is

\[
x(t) = x_s(t) + x_h(t)
\]

\[
= \frac{f}{\tilde{p}} \frac{e^{i(\omega t + \phi)}}{-\omega^2 - \omega_0^2} + A e^{\alpha_1 t} + B e^{\alpha_2 t}
\]

When \( \gamma > 0 \), homogeneous solution decays, so often long time,

\[
x(t) \rightarrow x_s(t) = \frac{f}{\tilde{p}} \frac{e^{i(\omega t + \phi)}}{-\omega \omega_0 (e^{\omega t} + e^{-\omega t})}
\]
Demonstration of driven oscillation.

How do you hear me? How does a system respond to external force?

To an applied scientist...

Air vibration

Mechanical vibrations

Light modulation

Wave oscillation

To an (arrogant) physicist, all these are nothing but

\[ \text{driving force} = m f \omega_0 (\omega t + \theta) \]

\[ x'' + \gamma x' + \omega_0^2 x = f \cos(\omega t + \theta) \]

- \( x \): displacement of anything
- \( \gamma \): friction/damping coefficient
- \( \omega_0 \): natural freq
- \( f \): external force
- \( \omega \): driving angular freq \( \frac{2\pi}{\text{period}} \)

Also \( Q = \frac{\omega_0}{\gamma} \) is the quality factor.
First we go to complex notation \[ \tilde{x} = x + iy, \quad \tilde{x} = \text{Re} \{\tilde{x}\}, \quad x \text{ and } y \]
satisfy
\[ \begin{align*}
\ddot{x} + y \dot{x} + \omega_0^2 x &= f \cos(\omega t + \theta) \\
\ddot{y} + y \dot{y} + \omega_0^2 y &= f \sin(\omega t + \theta)
\end{align*} \]
\[ x, y, f \in \text{real} \]

Combine the 2 eqns, we get
\[ f = fe^{i\theta} \]

\[ \ddot{\tilde{x}} + \gamma \dot{\tilde{x}} + \omega_0^2 \tilde{x} = f e^{i(\omega t + \theta)} = f e^{i\omega t} \]

The physics solution will be \[ \tilde{x}_h = \text{Re} \{\tilde{x}\} \]
\[ \tilde{x} = Ae^{-\omega t} \quad \Rightarrow \quad A \left( -\omega_0^2 + i\gamma \omega + \omega_0^2 \right) = f \]
See Lecture 2. Let, the general solution is
\[ \tilde{x} = \tilde{x}_s + \tilde{x}_h \]
\[ \begin{align*}
\text{Homogeneous solution } &\tilde{x}_h = Ae^{-\omega_0 t} + Be^{i\omega t} \\
\text{special solution } &\tilde{x}_s = \frac{f}{\omega_0^2 - \omega^2 + i\gamma \omega} e^{i\omega t}
\end{align*} \]

Real general solution \[ x = \text{Re} \{\tilde{x}\} = \text{Re} \{\tilde{x}_s\} + \text{Re} \{\tilde{x}_h\} \equiv x_s + x_h \]

The special solution is \[ x_s = \text{Re} \left[ \frac{f e^{i\omega t}}{c} \right] , \]

The driving freq \[ c = \omega^2 - \omega_0^2 + i\gamma \omega \]
\[ \equiv \rho e^{i\alpha} \]
\[ \Rightarrow \]
\[ x_s = \text{Re} \left[ \frac{f e^{i\omega t}}{\rho e^{i\alpha}} \right] = \frac{f}{\rho} \text{Re} \left[ e^{i(\omega t - \alpha)} \right] \]
\[ = A \cos(\omega t - \phi) \quad \phi = -\alpha \]
Amplitude \[ A = \frac{f}{\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}} \]

Phase shift \[ \phi = -\tan^{-1} \frac{\gamma \omega}{\omega_0^2 - \omega^2} \]
- $Q$ is set to zero in the textbook, we will follow it now $Q=0$.
- Both amplitude and phase depend on driving freq $\omega$ relative to $\omega_0$.

How do $A$ and $\phi$ look?

**Quality factor**

$$Q = \frac{\omega_0}{\gamma}$$

**Resonance**

$$Q = \infty \quad \gamma = 0$$

**Key features:**

- Sharp resonance $\omega = \omega_0$
- For high $Q$ oscillator
- Constant response at $\omega \rightarrow 0$
- Vanishing response $\omega \rightarrow \infty$
- Infinite amplitude
- On resonance when $Q = \infty$ (SHO)

**Natural** $\omega = \omega_0$

**Phase lag**

$$\phi$$

**Key features**

- Low driving freq $\omega \rightarrow 0$: no phase shift
- Resonance freq $\omega = \omega_0$: $\pi/2$ phase lag $\phi = -\pi/2$
- High freq limit $\omega \gg \omega_0$: $\pi$ phase lag $\phi = -\pi$
- Higher $Q$ $\Rightarrow$ sharper transition near resonance
- Infinite $Q = \infty$ (SHO) $\Rightarrow$ phase jumps at $\omega = \omega_0$ by $-\pi$
Let's look at the boring regimes

**Case 1**: \( w \ll w_0 \)  
low driving freq

\[
\lim_{w \to 0} A = \lim_{w \to 0} \frac{f}{w} \implies \lim_{w \to 0} A = \frac{f}{w_0} \quad \text{Hooke’s law}
\]

\[
\lim_{w \to 0} \phi = 0
\]

This is static force response.

\[
F = m f = k x \implies x = \frac{f}{k w_0^2}
\]

\[
\text{Indep of quality of oscillator}
\]

\[
\Rightarrow \text{nothing responds to infinite fast modulation!}
\]

**Case 2**: \( w \gg w_0 \)  
high driving freq

\[
\lim_{w \to 0} A = \lim_{w \to 0} \frac{f}{w^2} \to 0
\]

\[
\lim_{w \to 0} \phi = -\pi
\]

No response at infinite freq.

Very important concept!!

Near resonant driving freq

**Case 3** near resonance \( w \approx w_0 \)

We define detuning \( \Delta = w - w_0 \ll w_0 \)

\[
\Rightarrow w + w_0 \approx 2w_0
\]

\[
\Rightarrow \left( w - w_0 \right)^2 + \gamma^2 w_0^2 \approx 4w_0^2 \Delta^2 + \gamma^2 w_0^2
\]

\[
\approx 4w_0^2 \Delta^2 + \gamma^2 w_0^2
\]

(You will see this 100x in physics.)

Peak @ \( w \approx w_0 \),

\[
A_{\text{peak}} = \frac{f}{w_0 \gamma} = \frac{f}{w_0^2 Q}
\]
Lecture 2-3  Energy transfer in a driven oscillator

Consider a turkey in a microwave oven, we heat the bird by driving the electrons in the H2O molecules with microwaves at 2.45 GHz.

\( \omega_0/2\pi = 2.45 \text{ GHz} \) is one major resonance in H2O.

60\% eff

1000 W

How do we calculate energy transfer?

\[ \text{System gains energy from driving force } F_{\text{ex}} \text{ as} \]

\[ \Delta E = \text{Work} = \vec{F}_{\text{ex}} \cdot \Delta \vec{x} \]

Power of the driving force

\[ P = \frac{\Delta E}{\Delta t} = \frac{\vec{F}_{\text{ex}} \cdot \Delta \vec{x}}{\Delta t} = \vec{F}_{\text{ex}} \cdot \vec{v} \]

In our driven oscillator, two external forces act on the mass.

\[ -kx \]

\[ \text{friction} = -b \vec{u} \]

\[ \text{driving} = F_{\text{ex}} \cos \omega t \]

Equation of motion

\[ x'' + \gamma x' + \omega_0^2 x = F_{\text{ex}} \cos \omega t \]

\[ \gamma = \frac{b}{m}, \quad f = \frac{F_{\text{ex}}}{m} \]

How do these forces transfer energy to the oscillator?
Energy transfer due to friction:

\[ P_{\text{friction}} = \int_{t}^{t+\Delta t} f_{\text{friction}} \, dt \quad \Rightarrow \quad f_{\text{friction}} < 0 \]

\[ \Rightarrow \text{friction always takes energy out of system} \]

Given the special solution:

\[ x(t) = \frac{f}{p} \cos(\omega \cdot t + \phi) \]
\[ x'(t) = -\frac{f}{p} \omega \sin(\omega \cdot t + \phi) \]

\[ P_{\text{friction}} = -m \nu \frac{f}{p^2} \omega \sin^2(\omega \cdot t + \phi) \leq 0 \quad \text{at all time} \]

Averaging over one cycle:

\[ \langle P_{\text{friction}} \rangle = -m \nu \frac{f}{p} \int_{0}^{\pi/2} \sin^2(\omega \cdot t + \phi) \, dt \]

Not surprisingly, friction dissipates energy, driven or not.

Energy transfer due to external driving:

\[ P_{\text{ext}} = F_{\text{ext}} \cdot \nu = m \frac{f}{p} \omega \sin(\omega \cdot t + \phi) \]

\[ \langle \sin^2(\omega \cdot t) \rangle \]

\[ \langle \sin^2(\omega \cdot t + \phi) \rangle \quad \text{obviously oscillating!!} \]

\[ \text{What do we do with this?} \]

\[ \tan \phi = \frac{-\nu \omega}{l} \]

\[ \phi = \tan^{-1} \left( \frac{-\nu \omega}{l} \right) \]

\[ \sin \phi = \frac{-\nu \omega}{l} \]

\[ \cos \phi = \frac{\sqrt{l^2 - \nu^2 \omega^2}}{l} \]

\[ \Rightarrow \text{average to zero} \]

\[ \langle P_{\text{ext}} \rangle = \frac{m f^2 \omega^2}{2 p^2} \times \frac{1}{2} \sin^2 \phi \]

\[ = - \langle P_{\text{friction}} \rangle \]
\[ \langle \text{Total power transfer} \rangle = \langle \text{Pen} \rangle + \langle \text{P friction} \rangle = 0 \]

\[ x = \frac{f}{P} \cos (\omega t + \phi) \]

**Steady state**

\[ E = \text{const.} \]

Finally, total energy in the system \( E_{\text{tot}} = \text{Kineti}c + \text{potential} \)

\[ E = \frac{1}{2} k x^2 + \frac{1}{2} m v^2 = \frac{m}{2} \frac{d^2 x}{dt^2} \]

\[ = \left( \frac{m \omega^2 f^2}{P^2} \right) = \text{const. of time} \]

Thus \( \langle \text{Pen} \rangle = -\langle \text{P friction} \rangle = E \frac{\gamma}{2} \)

\[ \frac{\partial E}{\partial t} = \frac{\gamma}{2} \frac{\partial E}{\partial t} \]

energy stored in the system

\[ \frac{\partial E}{\partial t} = \frac{\gamma}{2} \frac{\partial E}{\partial t} \]

\[ \text{energy input from driving} \]

\( \Rightarrow \) the lower \( \gamma \) is the more energy is stored, less energy dissipated and higher quality!!

\[ Q = \frac{W_0}{\gamma} \]

See how \( E \) looks like

\[ E = m f \frac{\omega^2}{P^2} = m f \frac{\omega^2}{(\omega^2 - \omega_0^2)^2 + \gamma^2 \omega_0^2} \]

\[ \times \text{independent} \left\{ \begin{array}{l} \omega < \omega_0 \cdot \frac{\omega^2}{\omega_0^2} \propto \omega^2 \\ \omega > \omega_0 \cdot \frac{1}{\omega^2} \end{array} \right\} \]

\[ \omega \approx \omega_0 \cdot \frac{1}{\sqrt{\gamma}} \]