Lecture 5-1 Wave Equation

Coupled oscillators in the continuum limit

\[ \Delta \text{ goes to zero, } m \rightarrow \rho A, \quad x_n(t) \rightarrow \varphi(x,t) \]

\[ m \frac{\partial^2 \varphi}{\partial t^2} = k \left[ \varphi(x+\Delta) - 2 \varphi(x) + \varphi(x-\Delta) \right] \]

\[ \Rightarrow m \frac{\partial^2 \varphi}{\partial t^2} = \kappa A \left[ \frac{\varphi(x+\Delta) - \varphi(x)}{\Delta} - \frac{\varphi(x) - \varphi(x-\Delta)}{\Delta} \right] \]

\[ \approx \kappa A \left( \frac{\partial^2 \varphi(x+\Delta)}{\partial x^2} - \frac{\partial^2 \varphi(x-\Delta)}{\partial x^2} \right) \]

\[ \approx \kappa A \frac{\partial^4 \varphi(x)}{\partial x^4} \]

\[ \Delta \rightarrow 0 \Rightarrow \rho \frac{\partial^2 \varphi}{\partial t^2} = \kappa A \frac{\partial^2 \varphi}{\partial x^2} \]

\[ \rho = m/\Delta \quad \text{linear density} \]

\[ \varepsilon = \kappa A : \text{Young's modulus (1D)} \]

What's Young's modulus?

\[ \frac{\text{force}}{\text{area}} = \frac{F}{A} \]

Total length \( L = N \Delta \)

Displacement \( \delta L = \frac{F}{k} \)

\[ \varepsilon = \frac{\text{stress}}{\text{strain}} = \frac{\text{Pressure}}{\text{Fractional deformation}} = \frac{F/A}{\delta L/L} = \kappa A \]

Example: Rubber \( \varepsilon = 0.01 \) GPa (1 Pa = N/m²)

Wood \( = 10 \times 10^6 \) Pa

Cu \( = 200 \)

Diamond \( = 1000 \) \( \Rightarrow \varepsilon = \) stiffness

Wave equation \( \varepsilon = \rho \frac{\partial^2 \varphi}{\partial t^2} \) or \( \varphi_{tt} = \frac{1}{\rho} \varphi_{xx} \)

\[ c = \sqrt{\frac{\varepsilon}{\rho}} \]

(by d'Alembert & Euler, 1746)
How do we solve it?

Old method: \( y(x,t) = A(x)e^{i\omega t} \)

\[ \begin{align*}
\Rightarrow \quad & \frac{\partial y}{\partial t} = i\omega y, \quad \frac{\partial^2 y}{\partial x^2} = -\omega^2 y, \quad \frac{\partial y}{\partial x} = A'e^{i\omega t}, \quad \frac{\partial^2 y}{\partial x^2} = A''e^{i\omega t} \\
\Rightarrow -\omega A'e^{i\omega t} &= \omega^2 A''e^{i\omega t} \\
\Rightarrow A'' &= -A' = -\frac{\omega}{\omega^2} A
\end{align*} \]

Next we assume \( A(x) = e^{ikx} \) \( \Rightarrow -k^2 = -\frac{\omega^2}{\omega^2} \Rightarrow \omega = kv \)

\( \Rightarrow \) Solution is \( y(x,t) = (Ae^{ikx} + Be^{-ikx})(Ce^{i\omega t} + De^{-i\omega t}) \)

or any equivalent forms

\( \Rightarrow \) General solution includes all superpositions with \( \omega = kv \)

\[ y = \sum_j (A_j e^{ik_j(x-ut)} + B_j e^{-ik_j(x-ut)}) + C_j e^{ik_j(x+ut)} + D_j e^{-ik_j(x+ut)} \]

Finally we need the B.C.s to determine the coefficients.

Say, the initial position and velocity of all oscillators \( y(x,t=0) \) is \( \partial_t y(x,t=0) \)

Next lecture on Fourier transform we will show all functions can be written in the form of \( f(x) = \int e^{-ikx} f(k) \)
Method 2: General solution of the form $\psi(x,t) = f(x-ut)$

Back to $\Delta^2 \psi = u^2 \Delta_x \psi$, we can introduce $u = v - \frac{1}{2}$

so as long as the function looks similar when we exchange $x$ and $u$ would work.

Let's check $\psi = f(x \pm ut)$

$\Delta^2 \psi = f''(x \pm ut)$

How about $f(x, t)$?

Let's check out $\psi = f(x-ut)$ for some general $f(x)$

The wave equation supports all functions that propagate in any direction, with any form at velocity $v = \frac{\sqrt{E}}{\rho}$. 
5.2 Summary of discrete and continuous waves.

For a string of \( n \) oscillators

\[
X_n = \sum_{j} A_j e^{-i(\omega_j + \omega_j t)} + B_j e^{-i(\omega_j - \omega_j t)} + C_j e^{-i(\omega_j + \omega_j t)} + D_j e^{-i(\omega_j - \omega_j t)}
\]

\[
\omega_j = 2 \omega_0 \sin \frac{m \pi}{N+1}
\]

\[
\omega_0 = \sqrt{k/m}
\]

Solutions are

\[
X_n = \sum_{m=1}^{N} A_m \sin \frac{m \pi x}{L} \cos(\omega_m t + \phi_m)
\]

\[
\Rightarrow X_0 = X_{N+1} = 0, \quad \omega_m = 2 \omega_0 \sin \frac{m \pi}{2(N+1)}
\]

Boundary conditions

\[
X_n(0), \quad X_n(L)
\]

\[
m = 1, \quad m = 2, \quad m = 3, \quad m = 5, \quad 5 \text{ particles}
\]

\[
5 \text{ modes}
\]

\[
\begin{align*}
\varphi(x, t=0) & \quad \Rightarrow \varphi(x, t=0) \\
\frac{\partial}{\partial t} \varphi(x, t=0) & \quad \text{position, velocity}
\end{align*}
\]

\[
\sin \frac{\pi x}{L} \quad \sin \frac{3\pi x}{L}
\]

\[
\sin \frac{2\pi x}{L} \quad \sin \frac{3\pi x}{L}
\]

\[
\text{infinite modes}
\]

A continuous string

\[
\varphi(x) = \sum_{m=1}^{\infty} A_m \sin \frac{m \pi x}{L} \cos(\omega_m t + \phi_m)
\]

\[
\varphi(x=0) = \varphi(x=L) = 0
\]

\[
K_m = \frac{m \pi}{L} \quad \omega_m = K_m \nu
\]

\[
\varphi(x, t=0) \quad \frac{\partial}{\partial t} \varphi(x, t=0)
\]
In particular, for sinusoidal functions \( y = \cos k(x - ut) = f(x - ut) \)

A particular time

\( x = x_c \)

A particular position

A wavelength \( \lambda \)

\( k = \frac{2\pi}{\lambda} \) : wave \# \( t = 0 \)

\( t = 1 \) \( x \)

A period \( T \)

\( \omega = \frac{2\pi}{T} \) : angular freq

Velocity \( u \) \( \frac{\lambda}{T} \) : \( \text{wavelength/period} \)

All kinds of waves: \( \cos(kx - wt) \) or \( \sin(kx - wt) \) called : sinusoidal travelling waves

\( \cos(kx + \omega t) = \frac{1}{2} \cos(kx - wt) + \frac{1}{2} \cos(kx + wt) \) : standing waves

\( \rightarrow \) solitary waves \( \rightarrow \) shock waves

Wave eqn \( \partial^2 y = \nu \partial^2 x y \) describes those that remain the wave form and travels at a particular speed \( \nu = \sqrt{\frac{\rho}{\rho}} \)

\( \nu \) is highest for higher modulus (stiffness) \& lower mass density

\( \text{Air} \rightarrow \rho \text{ small} \frac{300 \text{ m}}{\text{s}} \)

\( \text{Steel} \rightarrow \rho \text{ large} \frac{1 \text{ km}}{\text{s}} \)
Consider we have $\Phi(x, t=0) = \phi(x)$, $\partial_t \Phi(x, t=0) = 0$. $\phi(0) = \phi(L) = 0$

$\Phi(x, t) = \sum_{m=1}^{\infty} \sin kmx (A_m \cos \omega_m t + B_m \sin \omega_m t)$

$\partial_t \Phi(x, t) = \sum_{m=1}^{\infty} \sin kmx (-A_m \omega_m \sin \omega_m t + B_m \omega_m \cos \omega_m t)$

$\Rightarrow \Phi(x) = \sum_{m=1}^{\infty} A_m \sin kmx$

$0 = \sum_{m=1}^{\infty} B_m \omega_m \sin \omega_m x$

How do we determine $A_m$ and $B_m$?

(Fourier Series)

Fourier realized that $\sin kmx$ and $\sin kmx$ are orthogonal.

Thus we can multiply $\sin \frac{nmx}{L}$ on both sides of the eqn, integrate

$\Rightarrow \int_0^L \Phi(x) \sin \frac{nmx}{L} \, dx = \sum_{m=1}^{\infty} A_m \int_0^L \sin \frac{nmx}{L} \sin \frac{nmx}{L} \, dx = \sum_{m=1}^{\infty} A_m \delta_{mn} \frac{L}{4}$

$= \frac{L}{4} A_n \Rightarrow A_n = \frac{L}{4} \int_0^L \phi(x) \sin \frac{nmx}{L} \, dx$

$\Rightarrow \int_0^L 0 \sin \frac{nmx}{L} \, dx = \sum_{m=1}^{\infty} B_m \delta_{mn} \frac{L}{4}$

$= \frac{L}{4} B_n \Rightarrow B_n = 0 \quad \forall n$

So we solved the problem

$\Phi(x, t) = \sum_{m=1}^{\infty} A_m \sin \frac{nmx}{L} \cos \frac{nm \omega_m t}{L}$

$A_n = \frac{4}{L} \int_0^L \phi(x) \sin \frac{nmx}{L} \, dx$