Lecture 9-1 Dispersion

There are 2 deep holes in the derivations in previous classes.

\[
\omega = \nu k \quad \text{is not true universally.}
\]

Example 1: Damped wave eqn

\[
(p^2 + \varepsilon^2)\phi = T\phi
\]

\[
\Rightarrow -\nu^2 \phi + \varepsilon \dot{\phi} + T k^2 \phi = 0
\]

\[
\omega = \frac{1}{\nu^2} \left( \varepsilon \pm \sqrt{\frac{4\nu^2 T k^2 - 4\varepsilon^2}{\nu^4}} \right)
\]

\[
= \frac{\varepsilon}{\nu^2} \pm \frac{\sqrt{T k^2 - \varepsilon^2}}{\nu^2}
\]

\[
= \pm \frac{1}{\nu} \sqrt{(1 - \frac{\varepsilon^2}{\nu^2}) k^2 + \frac{\varepsilon}{\nu^2}}
\]

\[
\Rightarrow \phi = e^{-\nu H} e^{ikx} e^{i\omega t}, \quad \omega = \sqrt{\frac{T k^2}{\nu^2} - \frac{\varepsilon^2}{\nu^4}}
\]

So what's the velocity of the wave?

Example 2: m •••• m

\[
\omega = 2\omega_0 \sin \frac{\Delta}{A}
\]

\[
\Rightarrow \phi(x,t) = \sin \left( \frac{x}{A} - \frac{\omega_0}{\Delta} t \right)
\]

\[
\Rightarrow \omega = 2\omega_0 \sin \frac{kA}{A} \sin \frac{\Delta}{A}
\]

\[
\Rightarrow \omega = \frac{\omega_0 \Delta k \equiv \nu k}{\text{small } k}
\]

\[
\Rightarrow \omega = \sqrt{T k^2} \quad \text{when } k \gg \frac{\varepsilon}{\nu^2}
\]

\[
\Rightarrow \omega = \frac{\varepsilon \nu}{k} \quad \text{when } k < \frac{\varepsilon}{\nu^2}, \text{ all damping}
\]

Strange thing happens when \[k = \frac{\varepsilon}{\nu^2}\]

double root ⇒ critical damping
So in general, we have \( \omega = \omega(k) \), called dispersion relation. What's exactly what's the propagation speed? See YouTube video: phase velocity \( v_p = \frac{\omega}{k} \) on abalone, group velocity \( v_g = \frac{d\omega}{dk} \).

Phase velocity: how \( \sin k \) propagates
\[
\sin(kx - \omega t) = \sin(kx - \omega t)
\]

Group velocity: how envelope propagates.

If \( \omega(k) \) is linear, \( \omega = kv \). \( \Rightarrow \) \( \frac{\omega}{k} = u \)

\( v_k \) is the ratio \( \frac{\omega}{k} \), \( v_g \) is the tangent \( \frac{d\omega}{dk} \)

You can find 4-5 different proofs for \( v_g \) in the textbook.

In typical situations, \( v_g < v_p \) what we perceive can be slower than the phase velocity \( v_p \).

\( v_g > v_p \) *Also mean not impossible to propagate > \( \sqrt{\omega / p} \)?
Proof: 

\[ \phi(x,t) = e^{ik_0x} \]

\[ = F(x) \times e^{iK_0x} = \int \varphi(k) e^{iKx} dk \]

\[ \Rightarrow F(x) = \int \varphi(k) e^{i(K-K_0)x} dk \]

\[ \Rightarrow F(k) = e^{-ik_0x} \varphi(k) \]

\[ \Rightarrow \varphi(x,t) = \int \varphi(k) e^{iKx} e^{-i\omega t} dk \]

\[ f(x-vt) = \int \varphi(k) e^{ik(x-vt)} dk \]

If \( \omega = \omega_k \) is linear

Now, \( \omega(k) = \omega(k_0) + \omega'(k_0)(K-K_0) + \ldots \)

\[ = \omega_0 + \omega'(k_0)(K-K_0) + \ldots \quad \omega_0 = \omega_k \]

\[ \varphi(x,t) \approx \int \varphi(k) e^{i(Kx-\omega_0t)} e^{-i\omega'(k_0)x} dk \]

\[ = e^{i(Kx-\omega_0t)} \int \varphi(k) e^{iKx-\omega'(k_0)x} dk \]

\[ \Rightarrow \varphi(x,t) = F(x-vt) e^{iK_0(x-vt)} \]

Note: the approximation was made, \( \omega(k) \) only taken to leading order, \( \Rightarrow F \) is a wide smooth function to grow higher order terms.
Quantum Mechanics: $\omega(cK) \rightarrow E(p)$
$E = \hbar \omega, \ p = \hbar k$

Classical
$u = \frac{d\omega}{dK} = \frac{dE}{dp} = \frac{d}{dp}\frac{p^2}{2m} = \frac{P}{m} = \frac{1}{\gamma}u_c$

Quantum
$u_p = \frac{\omega}{K} = \frac{E}{p} = \frac{\gamma m^2 c^4 + p^2 c^2}{p} > \frac{1}{\gamma}u_c$
Lecture 9-2  Doppler Effect and shock waves

Among all miscellaneous wave phenomena in Chapters 6 and 7, these are the most interesting and useful.

Doppler effect: How we know the age of universe, and about big bang.

Doppler Radar: Meteorology, navigation, police, siren,

Laser cooling, tests of fundamental symmetry ...

Crazy physics: sonic boom, shock waves, light boom, Cherenkov radiation;

Bottom line:

Stationary musician  Running musician  Sound speed musician

Play 1.5x  all music in a sh*t.

Laplacian supersonic musician!!  music played backwards.

Wait, we have not learned waves in 2D!!

$$\nabla^2 \psi = \frac{1}{v^2} \left( \nabla_x \cdot \nabla_x + \nabla_y \cdot \nabla_y + \nabla_z \cdot \nabla_z + \cdots \right) \psi$$

$$\equiv \nabla^2 \psi$$

Plane wave: $$\psi = e^{i(k \cdot \mathbf{x} - \omega t)} \Rightarrow \omega = \frac{v}{p} (k_x, k_y, k_z, ...)$$

Spherical waves: $$\frac{1}{r^2} \frac{\partial}{\partial r} (r \psi) \Rightarrow \psi = \frac{1}{r} e^{i(k r - \omega t)}$$

because energy spreads out, weaker afar.
Shock waves

\[ \omega = 0 \]
\[ \frac{\omega}{c} = \pm \frac{U_p}{c} \]

\[ U = U_p \quad \text{hear nothing} \]
\[ U = -U_p \quad \text{bass} \rightarrow \text{soprano} \]

\[ V = \pm U_p \]

When some moves faster

\[ \omega \rightarrow \infty \quad \text{sonic/flight boom} \]
\[ U = \frac{U_p}{c} \quad \omega = \frac{U}{c} \quad \text{soprano} \rightarrow \text{bass} \]

It's not hard to show

\[ \sin \theta = \frac{U_p}{V_s} \leq 1 \]

Related phenomena: wake

\[ \text{Supersonic coil} \]
Let's consider a point source

- A full oscillation is released every period by the source
  
  \[ \omega = \frac{2\pi}{T_0} \]
  
  \[ \omega_0 = \frac{2\pi}{T_0} \]

- Forward direction, \( \theta = 0 \)
  
  \[ \frac{\lambda}{\lambda_0} = 1 - \frac{v_p}{c} \]
  
  \[ \omega = \omega_0 \left( 1 + \frac{v_p}{c} \right) \]

- Backward direction, \( \theta = \pi \)
  
  \[ \frac{\lambda}{\lambda_0} = 1 + \frac{v_p}{c} \]
  
  \[ \omega = \omega_0 \left( 1 - \frac{v_p}{c} \right) \]

- In general:
  
  \[ \omega = \frac{v_p}{v_p - v_0 \cos \theta} \omega_0 \]

- The Doppler effect for a moving source

- Audience running toward source
  
  \[ T = \frac{\lambda_0}{v_0 + u} = \frac{T_0 (1 + \frac{v_p}{u})}{v_p} \]
  
  \[ \omega = \omega_0 \left( 1 + \frac{v_p}{u} \right) \]

- Audience runs away:
  
  \[ \omega = \omega_0 \left( 1 - \frac{v_p}{u} \right) \]

- Blue shift
  
  \[ \omega > \omega_0 \]

- Red shift
  
  \[ \omega < \omega_0 \]

- What if both are running?
  
  \[ \omega = \omega_0 \cdot \frac{v_p}{v_p - 2u_0 \cdot v_0} \]

- Audience speed
  
  \[ v_p = \hat{v}_p \cdot v_0 \]

- Source speed
  
  \[ v_p = \hat{v}_p \cdot v_0 \]

- Direction of sound propagation