Physics 238: Atomic Physics
Fall Quarter 2021
Problem Set #4
Due: 12:20 pm, Tuesday, November 9. Please submit in class.

1. **Bose-Einstein condensation (BEC)**

In the class we defined the condition for BEC as there are 1 or more particles in the ground state

\[ N_0 \geq 1, \text{ and the total particle number is } N = \sum_{i=0} N_i, \text{ where } N_i = N_0 e^{-\frac{E_i}{kT}} \text{ is the population in the } i\text{-th lowest quantum state of the system, and the ground state energy is set to } E_0 = 0. \]

In a large box of volume \( V = L^3 \), we have \( E_i = \frac{p_i^2}{2m} \), where \( m \) is the atomic mass. We can rewrite the sum as the integral as \( N = N_0 \int e^{-\frac{E}{kT}} \rho(E) dE \), where \( \rho(E) = \frac{2\pi^2}{h^3 \sqrt{2\pi m^3 V}} \) is the density of state in the box.

A. Complete the integral and show that the BEC condition can be written as \( N_0 = n \lambda_{dB}^3 \geq 1 \), where \( n = \frac{N}{V} \) is the total particle density and the expression defines the thermal de Broglie wavelength \( \lambda_{dB} \). Show that BEC occurs when \( k_B T \leq k_B T_c \equiv \frac{\hbar^2 n^2}{2\pi m} \), where \( T_c \) is the BEC critical temperature.

B. Typical BEC experiments adopt magnetic or optical traps which offer a conservative harmonic potential we model as \( V(r) = \frac{1}{2} m \omega^2 r^2 \) with \( \omega \) the trap frequency. We will explore whether the BEC condition depends on whether the trap is a box or a harmonic trap.

Given the eigen-energies of an atom in the harmonic trap is \( E = \hbar \omega (n_x + n_y + n_z) \), where \( n_i = 0,1,2 ... \) you can derive \( \rho(E) = \frac{E^2}{2\hbar^2 \omega^3} \). Show that \( N_0 = N \left( \frac{\hbar \omega}{k_B T} \right)^3 \) and BEC occurs when \( k_B T \leq k_B T_c \equiv N^\frac{1}{3} \hbar \omega \).

C. It appears that we have different expressions for the critical temperature, but they correspond to the same physical condition. Consider the BEC in the harmonic trap at the critical temperature \( T_c = k_B^{-1} N^\frac{1}{3} \hbar \omega \), show that the atomic density at the trap center \( n(r = 0) \) satisfies the BEC condition derived in A, namely,

\[ k_B T_c = \frac{\hbar^2 n(0)^2}{2\pi m}. \]

This result offers an important insight that BEC occurs at the center of the trap. Moreover, the condensation is a local quantum phenomena insensitive to the global trap geometry even so the calculation indicates.)

(Hint: A thermal gas in a harmonic trap is normally distributed with density \( n(r) = n(0)e^{-r^2/2\sigma^2} \), where the root-mean-square size of the sample \( \sigma = \sqrt{\langle r^2 \rangle} \) follows equipation theorem \( \frac{1}{2} m \omega^2 \sigma^2 = \frac{1}{2} k_B T \).
2. Wavefunction of an interacting Bose-Einstein condensate.

We start with non-interacting bosonic particles prepared in the ground state of a potential well $V(\vec{x})$, its wavefunction can be obtained by the standard Schrödinger equation

$$\left[ \frac{p^2}{2m} + V(\vec{x}) \right] \psi(\vec{x}) = E_0 \psi(\vec{x}).$$

The above is generalized for interacting bosons in the ground state by introducing the interaction term $U = gn$, which, for typical BECs, is dominated by two-body collision term with coupling constant $g$, $n(\vec{x}) = |\phi(\vec{x})|^2$ is the density of the sample. The “many-body” wavefunction $\phi(\vec{x})$ satisfies the mean-field Gross-Pitaevskii (GP) equation

$$\left[ \frac{p^2}{2m} + V(\vec{x}) + g|\phi(\vec{x})|^2 \right] \phi(\vec{x}) = \mu \phi(\vec{x})$$

and is normalized to the total particle number as $N = \int |\phi(\vec{x})|^2 d\vec{x}$. Here the constant $\mu$ here is called the chemical potential of the BEC.

A. Thomas-Fermi approximation

The GP equation has no general analytic solution, but a good approximation applies to most experiments where the solution is simple, that is, when the kinetic energy term $\frac{p^2}{2m}$ is negligible compared to the potential and interaction terms. Here the omission of the kinetic energy term, called the Thomas-Fermi approximation, is valid when the trap is macroscopic in size. Show that under this approximation we have

$$n(\vec{x}) = \begin{cases} \frac{1}{g} [\mu - V(\vec{x})] & V(\vec{x}) < \mu \\ 0 & V(\vec{x}) > \mu \end{cases}$$

B. Density profile of a harmonically trapped BEC

Let’s consider a generic case of N interacting atoms cooled to the ground state of a spherical harmonic trap $V(\vec{x}) = \frac{1}{2} kr^2$ with interaction strength $g$. Show that the density profile is

$$n(r) = \begin{cases} \frac{\mu}{g} \left( 1 - \frac{r^2}{R^2} \right) & r < R \\ 0 & r \geq R \end{cases}$$

where $R$ is the Thomas-Fermi radius of the BEC. Derive the chemical potential and the radius

$$\mu = \left( \frac{15}{8\pi} Ng \right)^{\frac{2}{5}} \left( \frac{k}{\mu} \right)^{\frac{3}{5}}$$

$$R = \left( \frac{15}{8\pi} Ng \right)^{\frac{1}{5}} \left( \frac{k}{\mu} \right)^{\frac{1}{10}}$$

C. Density profile of a BEC in a box

What is the density profile and chemical potential of a BEC with N atoms confined in a large square well potential with volume V?