1. Low energy excitations of a Bose-Einstein condensate

A Bose-Einstein condensate (BEC) can be described by the Gross-Pitaevskii equation under mean-field approximation,

\[ i\hbar \partial_t \psi(x, t) = \left( \frac{p^2}{2m} + V(x) + g|\psi(x, t)|^2 \right) \psi(x, t) \]

For a time-independent wavefunction with chemical potential \( \mu \) we have

\[ \left( \frac{p^2}{2m} + V(x) + g|\psi(x, t)|^2 \right) \psi(x, t) = \mu \psi(x, t) \]

Here \( p = -i\hbar \partial_x \) and we consider the condensate is confined in a large box \( V(x) = 0 \) with a uniform density \( n \). The ground state wavefunction is thus \( \psi_0(x, t) = \frac{n}{\sqrt{2\pi \hbar}} e^{-i\mu_0 t/\hbar} \), and the chemical potential is \( \mu_0 = gn \).

A. Here we consider low energy excited states of the system, assuming the BEC is weakly perturbed. The wavefunction can be expanded as \( \psi = \psi_0 + \epsilon \psi_1 \), where \( \epsilon \ll 1 \). Up to first order in \( \epsilon \), show that \( \psi_1 \) satisfies

\[ i\hbar \partial_t \psi_1 = \left( \frac{p^2}{2m} + 2gn \right) \psi_1 + g\psi_0^2 \psi_1^*, \]

which represents 2 coupled linear differential equations for \( \psi_1 \) and \( \psi_1^* \).

B. Apply the ansatz \( \psi_1 = e^{-i\mu_0 t/\hbar} [ue^{i(kx-\omega t)} + ve^{-i(kx-\omega t)}] \), where \( u \) and \( v \) are constant amplitudes of the plane waves, and show that they satisfy

\[ \begin{pmatrix} \frac{\hbar^2 k^2}{2m} + \mu_0 - \hbar \omega & \mu_0 \\ \mu_0 & \frac{\hbar^2 k^2}{2m} + \mu_0 + \hbar \omega \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0 \]

C. For solutions with non-zero amplitudes \( \mu, v \neq 0 \) show that the frequency and wave number are linked by the following dispersion \( \omega(k) \)

\[ \omega = \frac{1}{\hbar} \sqrt{\frac{\hbar^2 k^2}{2m} \left( \frac{\hbar^2 k^2}{2m} + 2\mu_0 \right)} \]

Thus the sound speed for long wavelength excitation \( k \to 0 \), is \( v = \lim_{k \to 0} \frac{\omega}{k} = \sqrt{\frac{\mu_0}{m}} \).
2. Low energy excitations of a Bose-Einstein condensate (second quantization)

In the second quantization form, we can write the energy of a bosonic system as

\[ H = \sum_k \varepsilon_k a_k^+ a_k + \frac{g}{2V} \sum_{k_1+k_2=k_3+k_4} a_{k_1}^+ a_{k_2}^+ a_{k_3} a_{k_4}, \]

where \( a_k^+ \) and \( a_k \) creates and annihilates a boson with momentum \( k \) and they satisfy the bosonic commutation relation \([a_k, a_{k'}^+] = \delta_{kk'}\). \( \varepsilon_k = \frac{\hbar^2 k^2}{2m} \) is the energy of a bare atom with momentum \( k \) and \( \frac{g}{V} \) is the interaction energy of a pair of atoms.

The wavefunction of the system can be described as \(|\psi\rangle = |n_1, n_2, ... \rangle\), where \( n_i \) is the population in the \( i \)-th lowest single atom eigenstate.

A. To gain some insight about the Hamiltonian, we assume there are only two momentum states \( k = \pm 1 \) in the system and there are only exactly 2 atoms that can occupy these states. The wavefunction can be a linear superposition of \(|n_{-1}, n_1\rangle \), \(|2,0\rangle \), \(|1,1\rangle \) and \(|0,2\rangle\), where \( n_{\pm 1} \) is the population in the \( k = \pm 1 \) state. Express the Hamiltonian as a matrix in the basis of the 3 states.

B. Now we consider the system where \( N_0 \) atoms are in the lowest momentum state \(|\psi_0\rangle = |N_0, 0, 0, ... \rangle\). Show that the energy of the system is \( <H> = \frac{g}{2V} N_0 (N_0 - 1) \equiv E_0 \).

C. Now we consider \( N_0 \gg 1 \) atoms in the zero momentum \( k = 0 \) state and few atoms \( N_i \ll N_0 \) in the finite momentum states. Total particle number is \( N = \sum_i N_i \). Approximating \( N_0 \pm 1 \approx N_0 \), show that we can approximate the system energy as

\[ H = E_0 + \sum_{k \neq 0} \varepsilon_k a_k^+ a_k + \frac{gN_0}{2V} \sum_{k \neq 0} (2a_k^+ a_k + a_k^+ a_{-k}^+ + a_k^+ a_{-k}+ a_k a_{-k}) \]

Remark: This result can be compared to the perturbation in 1A.

D. The Hamiltonian mixes states with \( k \) and \(-k\). Show that the following Bogoliubov transformation:

\[ a_k = u_k a_k + v_k a_{-k}^+ \]
\[ a_k^+ = u_k a_k^+ + v_k a_{-k} \]

Show that with suitable choice of the coefficients \( u_k \) and \( v_k \) can diagonalize the Hamiltonian as

\[ H = \text{const.} + \sum_{k \neq 0} \varepsilon_k a_k^+ a_k, \]

where \( a_k^+ \) and \( a_k \) creates and annihilates a bosonic quasi-particle with momentum \( k \).

Since the Hamiltonian is diagonal in their population \( a_k^+ a_k \). These quasi-particles are effective long-lived free particles in the system, called phonons that carry sound waves and do not interact with each other.

Remark: Compare your result to 1B and 1C.