Physics 143b: Honors Waves, Optics, and Thermo

Spring Quarter 2021

Problem Set #1

Due: 11:59 pm, Thursday, April 8. Please submit to Canvas.

1. **MATH** Please solve the following differential equations (5 points each)

   (a) $x'' + 4x' + 3x = 0$ with initial condition $x(0) = 1$ and $v(0) = 1$.
   (b) $x'' + 2x' + 5x = 0$ with initial condition $x(0) = 0$ and $v(0) = -1$.
   (c) $x'' + 2x' + 5x = \sin t$ with initial condition $x(0) = -1/10$ and $v(0) = 0$.
   (d) $x' + 2x = \cos t$ with initial condition $x(0) = 1$.
   (e) A particle with mass 1 is trapped in the local minimum of the potential $V(x) = \frac{x-1}{x^2+3}$. Derive the general solution of the particle's motion $x(t)$ in the vicinity of the minimum?
   (f) Determine the asymptotic behavior of $f(x, y) = x^{3/2}(x + 4y)^{1/2} - (x + y)^2$ when $x \gg y$.

2. **Damping an oscillator** (8 points each)
   A high-quality factor oscillator is the one that damps slowly and will continue to ring even after you stop driving it. (Think about a high-quality music instrument.) One way to quickly damp out its motion is to connect it to a strong damper (like what the piano pedal does). It is an art to choose the right damper to dissipate the oscillation fast. Here we assume the oscillator follows the underdamped oscillator model $x'' + \gamma x' + \omega_0^2 x = 0$ with $\gamma \ll 2\omega_0$ and it oscillates with a non-zero amplitude initially.

   At *t=0*, we suddenly apply a damper which increases the damping coefficient to $\gamma^* \gg \gamma$.

   (a) A smart choice is to select the damper with a critical damping coefficient $\gamma^* \approx 2\omega_0$. Show that the time scale for total energy to decay by *e* is given by $1/2\omega_0$.
   (b) Why shouldn't an even larger damping, say, $\gamma^* \gg 2\,\omega_0$ yield faster damping? Consider the case $x(0) = 1$ and $x'(0) = 0$ and evaluate the potential energy $V = m\omega_0^2 x^2/2$ and kinetic energy $E_k = mx'^2/2$. Show that in the large damping limit we have

$$E_k(t) \propto e^{-\mu t}$$

$$V(t) \propto e^{-\mu t},$$

   and both energies decay slowly at the rate of $\mu \approx \frac{2\omega_0^2}{\gamma^*}$.
   (c) Consider a simple example of a marble rolling in a bowl filled with air, water or honey, use your own words to explain why the energy of the marble decays slower when damping is very large?

3. **Resonant energy transfer** (8 points each)
A high-Q oscillator can store an immense amount of energy. Let's consider the extreme case of a harmonic oscillator with negligible friction (thus $Q \to \infty$) and natural frequency ω_0.

(a) If we drive the oscillator with the external force $f cos\ \omega t$ at a frequency very close to the resonance $\omega = (1 - \epsilon)\ \omega_0$ with $0 < \epsilon \ll 1$. Show that the total energy of the oscillator $E = E_k + V$ after the steady state is reached is

$$E_{max} \approx \frac{mf^2}{8\omega_0^2\epsilon^2},$$

Approaching resonance, the energy of the oscillator diverges.

(b) If the oscillator is at rest initially $x(0) = x'(0) = 0$ and we start applying the driving force $cos(\omega t)$ at t=0, show that with $0 < \epsilon \ll 1$ the total energy averaged over 1 cycle increases from zero quadratically as

$$< E(t) >= \frac{1}{8} mf^2t^2$$

(c) Combine the above results, show that it takes a long time to "charge up" a high-Q oscillator for small detuning $0 < \epsilon \ll 1$

$$T \sim \frac{1}{\omega\epsilon}$$

(Hint: you may expand $\epsilon \ll 1$ to leading order.)

4. **Chandelier as a seismometer? (8 points for (a) and (b), 6 points for (c))**
A chandelier is hung 1 m below the ceiling, and it weights 5kg. The damping coefficient is $\gamma = 0.01/s$. Local gravity is $g = 10\ m/s^2$.

(a) Determine the natural frequency ω_0 and quality factor $Q = \omega_0/\gamma$ of the chandelier.

(b) An earthquake hits and the building is trembling at the frequency near ω_0. You observe the chandelier starts oscillating with an amplitude as large as 3 cm as a result. Estimate how much the ceiling is displaced horizontally by the earthquake?

(c) Would you consider the chandelier an effective seismometer?