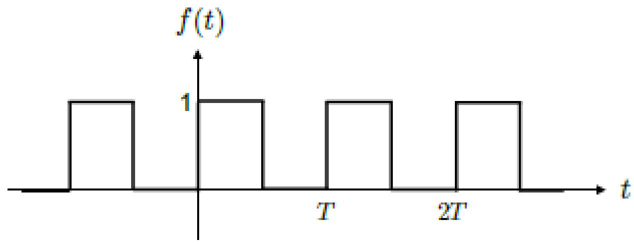
Physics 143b: Honors Waves, Optics, and Thermo

Spring Quarter 2021
Problem Set #3
Due: 11:59 pm, Thursday, April 22. Please submit to Canvas.

1. **Fourier series and transforms of a square wave** (6 points each)
   Consider a periodic function $f(t) = f(t+T)$, where T is the period. See below



We will expand it in 3 ways and compare the results.

(a) Expand $f(t)$ with the trigonometric functions with the same period

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos\frac{2\pi nt}{T} + b_n \sin\frac{2\pi nt}{T}).$$

Determine a_0, a_n and b_n.

(b) Expand $f(t)$ with the exponential series with the same period

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{i2\pi nt}{T}}.$$

Determine c_n.

(c) Fourier transform $f(t)$ over the entire range

$$f(t) = \int \tilde{f}(\omega) e^{i\omega t} d\omega.$$

Determine and determine $\tilde{f}(\omega)$.

(d) All these expansions are equivalent. Show that in general c_n and $\tilde{f}(\omega)$ can be expressed in terms of a_0, a_n and b_n.

Hint: You may use the following orthogonality conditions to determine the coefficients.

$$(a) \int_0^L \cos\left(\frac{2\pi nx}{L}\right) \cos\left(\frac{2\pi mx}{L}\right) dx = \int_0^L \sin\left(\frac{2\pi nx}{L}\right) \sin\left(\frac{2\pi mx}{L}\right) dx = \delta_{mn}\frac{L}{2}$$

$$\int_0^L \sin\left(\frac{2\pi nx}{L}\right) \cos\left(\frac{2\pi mx}{L}\right) dx = 0$$

$$(b) \int_0^L e^{\frac{i2\pi nx}{L}} e^{-\frac{i2\pi mx}{L}} dx = L\delta_{mn}$$

$$(c) \int_{-\infty}^{\infty} e^{ikx} e^{-ik'x} dx = 2\pi\delta(k-k')$$

2. **Fourier expansion exercise (6 points each)**
   Determine Fourier series of the following functions
   (a) $y(x) = |\sin x|$
   (b) $y(-\pi < x < \pi) = x$ and has a period of 2π, namely, $y(x + 2\pi) = y(x)$.
   (c) $y(x) = \begin{cases} \sin x, \sin x < 0 \\ 0, \ \sin x \geq 0 \end{cases}$
   (d) $x(t) = \int \tilde{x}(\omega) e^{i\omega t} d\omega$ is the particular solution of the driven harmonic oscillator equation $x''(t) + \gamma x'(t) + \omega_0^2 x(t) = f(t)$. Show that

$$\tilde{x}(\omega) = \frac{f(\omega)}{\omega_0^2 - \omega^2 + i\gamma\omega}$$

   where $\tilde{f}(\omega)$ is the Fourier transform of $f(t)$.

3. **General solution of a driven harmonic oscillator (6 points each)**
   (a) Following question 2., show that the particular solution of a harmonic oscillator driven by a general external force is formally given by

$$x(t) = \frac{1}{2\pi} \int \int \frac{f(\tau) e^{i\omega(t-\tau)}}{\omega_0^2 - \omega^2 + i\gamma\omega} d\omega d\tau.$$

   (b) As an example, we consider the external force $f(t)$ as described in question 1. Determine the explicit form of the solution of the oscillator.
   Hint: $x(t) = \frac{1}{2\omega_0^2} + \frac{1}{\pi} \sum_{n=1,3,5\ldots} \frac{1}{n} Im[\frac{e^{i\omega_n t}}{\omega_0^2 - \omega_n^2 + i\gamma\omega_n}]$, where $\omega_n = \frac{2\pi n}{T}$.

4. **Fourier Sine series** (6 points each)
   Here we consider a special case where the domain of the function $y(x)$ is only given in the range of $0 \leq x \leq L$. This occurs in (mechanical, sound, electromagnetic) waves that are spatially confined. Examples are violin strings and optical cavities. The function might not be periodic.

   (a) By virtually extending the domain of the function such that the function becomes odd, $y(-x) = -y(x)$, and periodic $y(x + 2L) = y(x)$, with period *2L*, show that the function can be expressed in terms of the Fourier Sine series

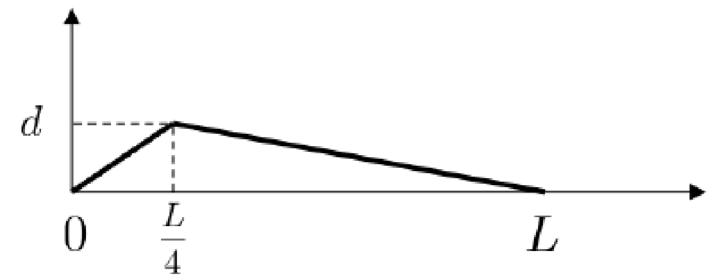
$$y(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

$$b_n = \frac{2}{L} \int_0^L y(x) \sin \frac{n\pi x}{L} dx$$

   (b) A guitar string of length L, fixed at both ends between x=0 and x=L, is displaced a small distance *d* at the point *L/4* at t=0. Find the Fourier Sine expansion of the amplitude. Evaluate the first 3 terms of the series

$$y(x) = \sum_{n=1}^{3} b_n \sin \frac{n\pi x}{L}$$

   (Hint: useful integrals: $\int dx \sin ax = -\frac{1}{a} \cos ax$, $\int dx\, x \sin ax = \frac{1}{a^2} \sin ax - \frac{x}{a} \cos ax$.)



5. Damped wave equation and guitar string (7 points each)

The guitar string moves according to the following damped wave equation

$$\rho\partial_t^2\psi(x,t) + b\partial_t\psi(x,t) = E\partial_x^2\psi(x,t)$$

With the boundary conditions

$$\psi(0,t) = \psi(L,t) = 0$$

(a) Determine the dispersion $\omega(k)$ of the eigenmodes, where ω is the angular frequency of the wave and k is the wavenumber.

(Hint: Dispersion is the relation between angular frequency and wave number. You may use the ansatz $\psi = Ae^{ikx}e^{i\tilde{\omega}t}$ and show that the eigenfrequency is $\omega = Re[\tilde{\omega}] = \sqrt{\frac{E}{\rho}k^2 - \frac{b^2}{4\rho^2}}$. Next show that to satisfy the boundary condition the wavefunction should have the form $\psi = A\sin kx\, e^{i\omega t}$ and the wavenumber $k = k_n = \frac{n\pi}{L}$ can only take discrete values with *n*=1,2,3….)

(b) Write down the general solution $\psi(x,t)$ for $t > 0$.
(Hint: The general solution is the linear superposition of all eigenmodes with allowable wavenumber $k = k_n$.)

(c) The initial position and velocity of the string is given by

$$\psi(x,0) = f(x), \qquad \partial_t\psi(x,0) = g(x)$$

Assuming $b = 0$ for simplicity, determine the wavefunction $\psi(x,t)$ for $t > 0$ and verify that the solution is real at all time.
(Hint: You will find the result of question 4 (a) useful.)

(d) A string instrument produces a rich spectrum of fundamental and overtones that please our ears. Ideally the pitch of the *n*-th overtone is (*n*+1) times the fundamental tone $\omega_n = (n+1)\omega_0$. Show that in the presence of small damping $0 < b \ll E\rho/L^2$, the overtones are slightly detuned by δ_n, which gives

$$\omega_n = (n+1)\omega_0 + \delta_n$$
$$\delta_n \approx \frac{1}{8}\frac{n(n+2)}{n+1}\frac{b^2}{\rho^2\omega_0}$$

Such effect is severe in the range of low frequency.