Physics 143b: Honors Waves, Optics, and Thermo

Spring Quarter 2021
Problem Set #4
Due: 11:59 pm, Thursday, April 29. Please submit to Canvas.

1. **Dirac's Delta function** *δ(x)* (10 points each)
   We may define the Dirac's Delta function based on the following procedure
   - $f(x)$ is any function that has an integrated area of $\int f(x)dx = 1$.
   - Dirac's delta function is defined as $\delta(x) = \lim_{\Delta\to 0}\frac{1}{\Delta}f\left(\frac{x}{\Delta}\right)$

   (a) A common choice of f by physicists is the Gaussian function $f(x) = \frac{1}{\sqrt{\pi}}e^{-x^2}$. Apply the above definition and prove that $\delta(x)$ satisfies the following properties
      1. $\delta(x \neq 0) = 0$
      2. $\delta(x = 0)$ diverges
      3. $\int \delta(x)dx = 1$
      4. $f(x) = \int f(u)\delta(x-u)du.$

   (b) Calculate the following
      1. $\int g(x)\delta(ax+b)dx$
      2. $\int g(x)\frac{d\delta(x)}{dx}\,dx, \quad g(\pm\infty) = 0$

2. **Kicked oscillator and Green's function (5 points each)**
   Let's consider a simple harmonic oscillator described by $x'' + \omega_0^2 x = f(t)$. The oscillator is at rest in the beginning $x(t=-\infty) = x'(t=-\infty) = 0$. At $t = t_0$, you kick the oscillator by an impulse. Immediately after the kick we have $x({t_0}^+) = 0$ and $x'(t_0^+) = v_0$.
   (a) Show that the impulse can be written as $f(t) = v_0\delta(t-t_0)$ and the solution is

$$x(t) = \begin{cases} 0, & t < t_0 \\ \frac{v_0}{\omega_0}\sin\omega_0(t-t_0), & t \geq t_0 \end{cases}$$

   This is the so-called Green's function of the differential equation.
   (b) Now we consider a general driving force $F(t)$, which, according to question 1 (a) 4., can be pictured as a summation of many kicks or delta functions, namely,

$$F(t) = \int F(\tau)\delta(t-\tau)d\tau.$$

   From the superposition principle, the solution is also the summation of the responses to individual kicks. Show that the general solution of the oscillator, initially at rest and then driven by an arbitrary force $F(t)$, is given by

$$x(t) = \int_{-\infty}^{t}\frac{F(\tau)}{\omega_0}\sin\omega_0(t-\tau)d\tau.$$

3. **Energy and energy flow in a wave (5 points each)**
Here we will investigate how waves transport energy in a medium (string, air, water…). Assume the wave (transverse or longitudinal) satisfies the following wave equation

$$\rho\partial_t^2\psi(x,t) = T\partial_x^2\psi(x,t),$$

where ρ is the linear density of the medium and T is the tension in the medium. A traveling wave is given by $\psi(x,t) = A\cos k(x - vt)$, where $v = \sqrt{T/\rho}$.

(a) Consider a small section between x and $x + \Delta x$, show that the energy densities are given by
kinetic energy density: $\rho_K = \frac{1}{2}\rho(\partial_t\psi)^2$
potential energy density: $\rho_U = \frac{1}{2}T(\partial_x\psi)^2$.
(Hint: Kinetic energy is given by $\frac{1}{2}mV^2$ of the section. As for potential energy, think about how potential energy $\frac{kx^2}{2}$ is derived by stretching a spring. Here how much does the tension T extend the length of the section?)

(b) Given the traveling wave $\psi(x,t)$, calculate the total energy densities ρ_K and ρ_U. Show that the energy is propagating. At some point the total energy of the section becomes zero $\rho_E = \rho_K + \rho_U$=0. Where does the energy go?

(c) Show the energy transfer per unit time is given by the
energy flux: $j_E(x,t) = -T\partial_x\psi\partial_t\psi$.
Determine the energy flux for the traveling wave.
(Hint: The energy that flows to the section comes from the work done by its neighboring sections through the tension force from $dW = T_y \cdot d\psi$, and use $\partial_x\psi = \frac{T_y}{T_x} \approx \frac{T_y}{T}$.)

(d) Show that for a traveling wave given by $\psi(x,t) = A\cos k(x \pm vt)$, we have

$$j_E = \pm v\rho_E.$$

Thus energy flux is the product of the energy density by the energy propagation velocity.

4. **Transmission and reflection of waves (10 points)**
Consider a wave moving toward an interface at $x = 0$, described by the equation

$$\partial_t^2 \psi(x,t) = v(x)^2 \partial_x^2 \psi(x,t),$$

where the wave propagation velocity changes across the interface

$$v = \begin{cases} v_L, & x < 0 \\ v_R, & x > 0. \end{cases}$$

Now consider an incident wave coming from the right side toward the interface

$$\psi_{in}(x > 0, t) = A \cos k(x + v_R t).$$

We may solve the equation with the following ansatz

$$\psi(x) = \begin{cases} Ae^{ik(x+v_R t)} + Be^{ik(-x+v_R t)}, & x > 0 \\ Ce^{ik^*(x+v_L t)}, & x < 0. \end{cases}$$

where B and C are the reflection and transmission amplitudes. At the end of the day, the solution is the real part of the ansatz.

(a) Determine B and C in terms of A and k^* using the boundary conditions.

(b) Determine the total energy flux of the incident, reflected and transmitted waves j_A, j_B and j_c and show that we have $j_A = j_B + j_C$ at the interface, promised by energy conservation. (Hint: you may assume the tension of the medium is T.)

(c) By flipping the time arrow $t \to -t$, the following should also be a solution

$$\psi(x) = \begin{cases} Ae^{ik(x-v_R t)} + Be^{-ik(x+v_R t)}, & x > 0 \\ Ce^{ik^*(x-v_L t)}, & x < 0. \end{cases}$$

However, here we have two waves with amplitudes B and C propagating toward the interface from both directions, but only one wave comes out?? Is such solution unphysical? Show that if you treat B and C as 2 independent incident waves, transmission of the B wave exactly cancels the reflection of the C wave, and thus no wave is propagating on left side $x < 0$ toward $x = -\infty$.

5. **Decibel scale of the strength of sound (5 points each)**
Alexander Bell, the inventor of phone, introduced the unit of bel, which became decibel in acoustics: Zero decibel (0 dB) is defined as strength of the sound wave that produces $\pm 20\ \mu Pa$ in the air pressure ($\mu Pa = 10^{-6} Pa$), which is also the typical limit of human hearing. Decibel is calculated in log scale such that every +20 dB corresponds to 10 times higher pressure. For instance, 20dB corresponds to 200 μPa and 40dB corresponds to 2 mPa. Human heating is damaged above 100dB, corresponding to 2 Pa.

(a) What is the assumed limit of human hearing? Calculate the intensity of a 1D acoustic wave at 0 dB.
(Hint: Intensity is energy delivered per area per time in the unit of Watt/m^2.)

(b) Show that sound cannot be louder than 200dB.

(c) How much is the maximum displacement $\psi(x)$ of air molecules away from equilibrium in the presence of acoustic waves at 0dB and 100dB at frequency = 100 Hz?

Material	Density (kg/m^3)	Compressibility (1/GPa)
Air	1.22	7200
Water	1000	0.5
Copper	8960	0.0073

(GPa = 10^9 Pa.)