Physics 143b: Honors Waves, Optics, and Thermo

Spring Quarter 2021
Problem Set #5
Due: 11:59 pm, Thursday, May 6. Please submit to Canvas.

1. **(Math) Vector calculus (15 points each)**
   a) A vector field $\vec{A}(x, y, z)$ carries a source if $\nabla \cdot \vec{A} > 0$ and a sink if $\nabla \cdot \vec{A} < 0$. It carries circulation if $\nabla \times \vec{A}$ is non-zero. Evaluate their divergence and curl and determine if the following fields carry source/sink or circulations?
   $\overrightarrow{A_1} = (\mathrm{e}^{\mathrm{y}}, 1, \mathrm{e}^{\mathrm{x}})$
   $\overrightarrow{A_2} = (z^2 x, 0, -\mathrm{z}\mathrm{x}^2)$
   $\overrightarrow{A_3} = (\cos y, \sin z - \sin x, -\cos y)$

   b) Prove the following vector identities (ϕ, ψ are scalar fields, $\vec{A}$ is a vector field.)
   $\nabla \cdot (\phi\vec{A}) = \phi\nabla \cdot \vec{A} + (\vec{A} \cdot \nabla)\phi$
   $\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2\vec{A}$
   $\nabla \cdot (\phi\nabla\psi - \psi\nabla\phi) = \phi\nabla^2\psi - \psi\nabla^2\phi$

2. **Propagation of a wavepacket (10 points each)**
   Given the dispersion $\omega(k)$ we will clarify the difference between phase velocity $v_p = \omega/k$ and group velocity $v_g = d\omega/dk$ with examples.

a) Consider the superposition of two waves $\psi = \cos(k_1 x - \omega_1 t) + \cos(k_2 x - \omega_2 t)$, where $\omega_1 = \omega(k_1)$ and $\omega_2 = \omega(k_2)$.
Prove that the wavefunction can also be described by a single traveling wave that propagates at the speed $v = (\omega_1 + \omega_2)/(k_1 + k_2)$ with an envelope function that propagates at the speed $v' = (\omega_1 - \omega_2)/(k_1 - k_2)$.

b) When the wavenumbers are similar $k_1 \approx k_2$ show that v becomes the phase velocity and v' becomes the group velocity.

c) Sound propagation in air is weakly dispersive $\omega(k) = vk(1 + \epsilon k)$. Let's compare the lowest piano key A_0 at $\omega_L = 2\pi \times 27.5$Hz and the highest key C_8 at $\omega_R = 2\pi \times 4186$ Hz. Their phase velocities are measured to be $v_L = 343.26$m/s and $v_R = 343.38$m/s. Determine ϵ in SI unit and their group velocities.

3. Doppler effect and echo (20 points)

Old-fashioned Doppler radar speed guns use ultrasound to measure the velocity. (Modern ones use microwaves). Assume the gun emits sound wave at frequency ω, the phase velocity is v_p and the police detects the frequency of the reflected waves as ω_r.

a) A car is moving straight away from the police at the speed $v < v_p$, show that the frequency of the reflected waves has a frequency of

$$\omega_r = \omega_0 \frac{v_p - v}{v_p + v}$$

b) In the presence of strong wind, the above formula needs further corrections. Assume the wind is in the same direction as the car and its speed is w, show that the reflected frequency is

$$\omega_\mathrm{r} = \omega_0 \frac{v_p - w}{v_p + w} \frac{v_p + w - v}{v_p - w + v}$$

Hint: Sound propagates in air. So when the air moves at velocity w, sound propagates faster in the same direction at velocity $v_p + w$, and slower in the opposite direction at $v_p - w$. Determine the time duration when two consecutive wavefronts are reflected by the car, and the spatial separation between them. The frequency of the echo here is determined by the duration the police receives two consecutive reflected wavefronts from the car.