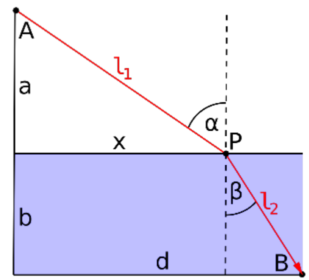
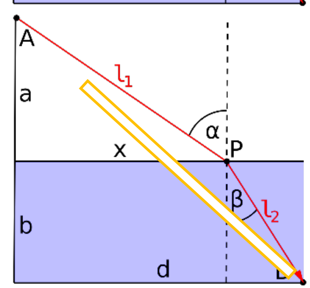
Physics 143b: Honors Waves, Optics, and Thermo

Spring Quarter 2021
Problem Set #6
Due: 11:59 pm, Thursday, May 13. Please submit to Canvas.

1. Fermat's principle (9 points each)
Snell's law $n_1 \sin\theta_1 = n_2 \sin\theta_2$ is a consequence of Maxwell's equations. However, Maxwell's equation was completed by Maxwell in 1861, Snell's law was discovered few centuries earlier by many, including Snellius (1580~1626) and P. Fermat in 1662. Here we will see how Fermat derived the law based on the *Fermat principle* of least time:
*"The path taken by a ray between two given points is the path that can be traversed in the **least time**."*



   a) Given the index of refraction of air $n=n_1$ and water $n=n_2$, a ray passes point A above the water and point B below the water. See figure for the dimensions. Show that when the travel time $T(x) = \frac{l_1(x)}{c/n_1} + \frac{l_2(x)}{c/n_2}$ is minimized, you get Snell's law $n_1 \sin\alpha = n_2 \sin\beta$.



   b) The above seems to suggest that Fermat principle is correct, right? Here is a thought experiment.
Let's stick a long straight straw from A to B and the straw is filled with air. A ray going straight from A to B along the straw will obviously reach B faster than along the red path described in a). According to the Fermat principle, light will take the fastest path and thus no light will ever follow the red path when you put in the straw. Is the above argument correct? Please explain.

2. Electromagnetic (EM) waves in conductors. (13 points each)
EM waves propagate differently in metal than in insulators since electrons in metal can move and contribute to the fields. We will investigate how EM waves propagate in an ohmic conductor with charge density ρ and current density $j = \sigma E$, where σ is the conductivity. Maxwell equations are

$$\nabla \cdot E = \frac{\rho}{\epsilon}$$
$$\nabla \cdot B = 0$$
$$\nabla \times E = -\partial_t B$$
$$\nabla \times \mathrm{B} = \mu\epsilon\, \partial_t E + \mu j$$

   a) Show that the electric field and magnetic field satisfy the wave equation:

$$\mu\epsilon\partial_t^2 E + \mu\sigma\partial_t E = \nabla^2 E - \frac{1}{\epsilon}\nabla\rho$$
$$\mu\epsilon\partial_t^2 B + \mu\sigma\partial_t B = \nabla^2 B$$

b) Assume there is no free charge $\rho = 0$, show that the solution of an EM wave propagating in the z direction can be written as

$E(z,t) = E_0 e^{-\frac{z}{z_0}} e^{ik(z-v_p t)}$,

where $z_0 \geq 0$ is called the skin depth and determines the how far the field can penetrate into the conductor, and v_p is the phase velocity of the wave. Show that they are given by

$$v_p = \frac{\omega}{k} = \sqrt{\frac{2}{\mu\epsilon(1+\sqrt{1+\frac{\sigma^2}{\omega^2\epsilon^2}})}}$$

$$z_0 = \frac{2}{v_p \mu \sigma}.$$

(Hint: You may consider the conductor occupies the space $z \geq 0$ and the light propagates toward the conductor from $z = -\infty$ toward the origin $z = 0$.)

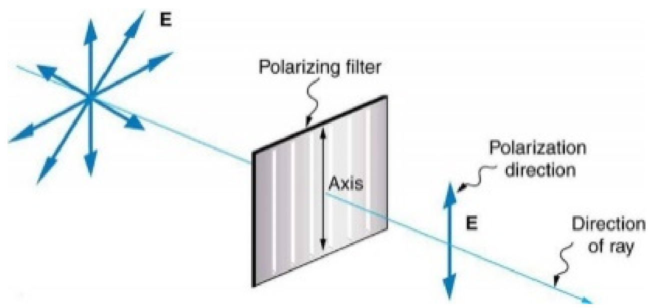
(Hint: You may assume the ansatz $E = e^{i(\tilde{k}z - \omega t)}$, where $\tilde{k} = k + i/z_0$ combines k and z_0 which may simplify your calculation. Other ansatzes should work just as well.)

c) The above result shows lots of weird properties about waves in conductors. First of all, the waves propagate in a conductor at a very different speed than in an insulator with the same μ and ϵ, where the velocity is $v_p = v_g = \frac{1}{\sqrt{\mu\epsilon}} \equiv \frac{c}{n}$. Show that the In a good conductor with conductivity $\sigma \gg \omega\epsilon$, the phase velocity is much slower than $\frac{c}{n}$ and is approximately $v_p \approx \frac{c}{n}\sqrt{\frac{2\omega\epsilon}{\sigma}}$ and the group velocity is about twice $v_g \approx 2v_p$. How fast does light (say, wavelength=500nm in vacuum) propagate in aluminum ($\sigma = 3.8 \times \frac{10^7}{\Omega\, m}, \epsilon \approx \epsilon_0$) and how deep can it penetrate into the metal?

d) Radiation intensity is given by the Poynting vector $< S(z,t) >= \frac{1}{\mu} < E(z,t) \times B(z,t) >_t = \frac{1}{2\mu v_p} E^2 = I_0 e^{-2z/z_0}$. Household microwave oven operates at about 1kW with frequency $\frac{\omega}{2\pi}$ =2 GHz and about 40 cm in size. Such radiation can in principle penetrates through the door and affects human. If the door is made of an aluminum foil of 0.1 mm in thickness. Estimate (order of magnitude) the leakage power and compare your result to the sunlight radiation and 1,367 W/m^2 and the FCC microwave regulation limit of 160W/m^2.

(Hint: Standard microwave oven doors use thicker metal mesh with thickness > 0.7 mm.)

3. Polarizers (10 points each)
Polaroid polarizing filters are made of nitrocellulose polynemer film, where the polymers form needle-like crystals that only absorb light with polarization (direction of electric field) along the direction of the crystals. The axis of the polarizer is typically defined as the direction that the light can transmit. See figure, where the polarizer axis is in the vertical (x-)direction, and the light propagates in z.



a) Assume the incident beam electric field is given by $E_i = (E_x, E_y)e^{ik(z-ct)}$, and the polarizer is aligned in the x direction as shown in the figure. The transmission rejects E field in the y-direction and thus the transmitted beam is $E_t = E_x(1,0)e^{ik(z-ct)}$. If we place a second polarizer rotated by $+\frac{\pi}{2}$ relative to the first one after the first one, show or argue that the transmission is zero.

b) Now we add a third polarizers between the two polarizers at an angle θ relative to the first one. Show that light can transmit again with electric field $E_i = E_x(0, \sin\theta\cos\theta)$.

c) Show that when an incident beam with electric field $E_{in} = (E_x^{in}, E_y^{in})$ passes through a polarizer rotated by $+\theta$ relative to the x-axis, the outgoing field is given by

$$\begin{pmatrix} E_x^{out} \\ E_y^{out} \end{pmatrix} = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix} \begin{pmatrix} E_x^{in} \\ E_y^{in} \end{pmatrix}.$$

(Hint: You may use superposition principle to determine the elements of the polarizer matrix.)