Physics 143b: Honors Waves, Optics, and Thermo

Spring Quarter 2021

Problem Set #8

Due: 11:59 pm, Thursday, May 27. Please submit to Canvas.

1. Ideal gas (10 points each)

   The laws we learned about the ideal gas are $PV = NkT$ and internal energy of the gas $U = \gamma NkT$, where $\gamma = 3/2$ for atomic gas and $\approx 5/2$ for molecular gas. We may investigate other properties about the ideal gas.

   a) Heat capacity:

      Given the isochoric specific heat $c_V = \frac{1}{M}\partial_T Q|_V$ we derived in Lecture 15-1, show that we can write the internal energy of the gas as

      $$U = c_V NmT,$$

      where m is the atomic/molecular mass.

   b) Compressibility:

      In the derivation of sound speed few weeks ago, we introduce the compressibility as the fractional reduction of the volume per unit pressure applied to the system:

      $$\beta = -\frac{1}{V}\frac{\partial V}{\partial P}$$

      We are now in good position to derive it. Show that the isothermal and isentropic compressibility of an ideal gas are $\beta_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T = \frac{1}{P}$ and $\beta_S = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_S = \frac{\gamma}{(\gamma+1)P}$.

      Here isothermal means the temperature is kept constant, while isentropic means the process is adiabatic and reversible. Which β should we use to estimate the speed of sound $v = 1/\sqrt{n\beta}$? It turns out isentropic compressibility β_S gives a better estimation. Give your argument why it is more reasonable to use β_S than β_T?

      (Hint: when sound wave shakes the molecules, will there be heat flowing between molecules? If yes, temperature could be a constant, if not, the process is adiabatic.)

   c) Thermal expansion:

      Thermal expansion determines the fractional change of the system when the temperature increases by 1 unit. There are again two possible processes: isobaric thermal expansion coefficient $\alpha_V = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P$ and isochoric coefficient $\alpha_p = \frac{1}{P}\left(\frac{\partial P}{\partial T}\right)_V$. Show that both lead to the same result as $\alpha_V = \alpha_P = \frac{1}{T}$.

      How much does the mean molecular distance in an ideal gas increases fractionally when temperature increases from 300K to 301K?

2. From ideal gas law to entropy (10 points each)
   Consider a system with energy U can exchange energy with the environment in two ways $\Delta U = \Delta Q - P\Delta V$ including heat exchange $\Delta Q = T\Delta S$ and mechanical work exchange $-P\Delta V$.
   a) An adiabatic process is the process that forbids heat exchange $\Delta Q = 0$. Assuming the system is described by the ideal gas law $PV = NkT$ and $U = \gamma NkT$, show that an adiabatic process is described by $PV^{(\gamma+1)/\gamma} = C$ and argue that the constant C is related to the entropy S.

   b) To understand the relationship between adiabatic process and entropy, we first try to determine entropy of an ideal gas. Use $dU = TdS - PdV$ and show that the entropy difference between the system in state 1 with pressure P_1, volume V_1 and in state 2 with pressure P_2, volume V_2 is given by

$$S_2 - S_1 = c_v Nm \ln \frac{P_2 V_2^{(\gamma+1)/\gamma}}{P_1 V_1^{(\gamma+1)/\gamma}}.$$

   Hint: Recast the equation as a differential equation $dS = \frac{dU}{T} - \frac{PdV}{T}$. Then use the ideal gas law to replace U in terms of P and V, and then integrate the differential equation.

   c) Show that this result is compatible with the Sackur-Tetrode equation for γ=3/2.

$$S = Nk \ln \frac{V}{N}\left(\frac{4\pi m}{3h^2}\frac{U}{N}\right)^{3/2}$$

   Hint: This derivation yields Planck constant h. One may only imagine that Mr. Dr. Sackur could have determined the value of h few decades before Max Planck.

   d) Determine the explicit form of the constant C in terms of the entropy S.

3. Maxwell-Boltzmann distribution (10 points each)
   A great insight from Maxwell is that the velocity distribution of molecules in an ideal gas with mass m and velocity $\vec{v} = (v_x, v_y, v_z)$ is given by $P(\vec{v}) = p(v_x)p(v_y)p(v_z)$, where $p(x) \propto e^{-mx^2/2kT}$ is the Maxwell-Boltsmann distribution
   a) Given the probability conservation condition $\int_{-\infty}^{\infty} p(x)dx = 1$ for a stochastic variable x, determine the explicit form of $p(x)$ and $P(\vec{v})$.
   b) Show that the probability distribution of P(v), where $v = |\vec{v}|$ >0 is the absolute value of the molecular velocity, is given by

$$\mathrm{p}(v) = \left(\frac{m}{2\pi kT}\right)^{3/2} 4\pi v^2 e^{-mv^2/2kT},$$

   which is the most common form of Maxwell-Boltzmann distribution.

Hint: A coordinate transform from Cartesian coordinate (x, y, z) to spherical coordinate should satisfy probability conservation. Assume the probability to find a particle in space within a small volume element dv is $dP = P(\vec{r})dv$, the same probability should be found in a new coordinate system, namely,

$$dP = P(\vec{r})dxdydz = P(\vec{r})r^2 \sin\theta \, drd\theta d\phi \equiv P(r, \theta, \phi)drd\theta d\phi.$$

e) Determine the root-mean-square velocity $v_{rms} = \sqrt{< v^2 >}$ and show that $v_{rms} = \sqrt{\frac{3kT}{m}}$ is consistent with the equipartition theorem. Compare the velocity to the sound speed, can one possibly heat up the gas enough such that the rms molecular velocity is higher than the sound velocity $v_p = \frac{1}{\sqrt{n\beta}}$?

4. Refrigeration (10 points each)
Refrigerator is a device that receives energy (electricity) from the Source of work (power plant) in order to extract energy from the Heat source (food in the fridge) at a lower temperature Tc and deliver the energy into the Heat sink (atmosphere) at a higher temperature T_H.

$$\begin{tikzcd}
\text{Heat sink } T_h & \text{Refrigerator} \arrow[l, "Q_h = W + Q_c"'] & \text{Source of work} \arrow[l, "W"'] \\
& \text{Heat source } T_c \arrow[u, "Q_c"'] &
\end{tikzcd}$$

a) Compare the above process with an engine, we find that the above process is essentially a Carnot cycle running in reverse. Determine how much energy W is demanded in order to extract one Joule of energy from the cold source.

Hint: Calculate $\frac{\Delta W}{\Delta Q_c}$ according to the definition in the diagram. You may just use the results we derived in Lecture 15-2 and assume the processes are now running backwards. Show the result in terms of the temperatures T_H and T_C.

b) Your fridge keeps the food at around 40 F, while the ambient temperature is 72 F. How many Joules of electricity are needed to remove 1 Joule of energy from the food in the fridge?

c) How far can we cool? One can now cool atoms to nano-Kelvins. Show that if we treat the atom as the heat source at T_C and the lab as the heat sink, the amount of energy required to cool an atom from $T_C = T_H = 300\ K$ to $T_c = 10^{-9}$ using the reverse Carnot cycle would be

$$W = c_v k[T_H \left(1 - \ln \frac{T_H}{T_C}\right) - T_C]$$

d) Show that you need infinite amount of energy to cool an atom from any finite temperature to zero temperature. The divergence goes logarithmically in the limit of $T_c = 0$ as $W = c_v k T_H \ln \frac{T_H}{T_C}$.

Hint: This result is a demonstration that zero temperature is not attainable.