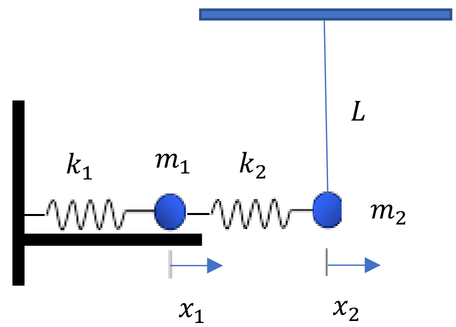
Physics 143b: Honors Waves, Optics, and Thermo

Spring Quarter 2021

MidTerm

Due: 11:59 pm, Saturday, May 1. Please submit to Canvas.

1. Coupled oscillators (12 points each)



Point mass m_1 on a frictionless horizontal plane is connected to a spring with force constant k_1. An ideal pendulum of length L with the bob of mass m_2 (gravitational force is $m_2 g$) couples to m_1 through a second spring with force constant k_2. In equilibrium their displacements are $x_1 = x_2 = 0$. Here we consider small amplitude oscillations $x_1, x_2 \ll L$ away from equilibrium in the following cases

(a) Given equal mass $m_1 = m_2 \equiv m$, equal force constant $k_1 = k_2 \equiv m\omega_0^2$ and the eigen-frequencies are found to be ω_0 and $\sqrt{3}\omega_0$. Determine L.

(b) Following (a), write down the general solution of the system and describe the motion of the two normal modes in words associated with the two eigenfrequencies.

(c) Following (a), an external force $F(t) = mf \sin \omega t$ applies only to point mass m_1. Determine the steady state motion of the two masses. (namely, the particular solution of the system.)

2. Sound tube (14 points each)

A sound tube with length of L accommodates sound waves that satisfy the wave equation

$$n\partial_t^2 \psi(x,t) = \frac{1}{\beta}\partial_x^2 \psi(x,t),$$

where n is the air density, β is the compressibility and the sound velocity is $v = (\beta n)^{-1/2}$. You sing into the tube at $x = L$ such that the air molecules at $x = L$ vibrate as $\psi(x = L, t) = f(t)$. The other end is sealed such that $\psi(x = 0, t) = 0$ at all times.

(a) Use the ansatz $\psi(x,t) = e^{ikx}e^{i\omega t}$, or any equivalent forms, and show that the general solution that satisfies the boundary condition $\psi(x = 0, t) = 0$ is given by

$$\psi(x,t) = \int A(k) \sin kx (A\, e^{ikvt} + Be^{-ikvt}) dk$$

(b) You single into the tube with a single tone $f(t) = f \cos \omega t$. Show that the wavefunction in the tube is given by

$$\psi(x,t) = c \sin\frac{\omega x}{v} \cos \omega t,$$

and determine the amplitude c.

(Hint: You may solve the equation assuming $f(t) = fe^{i\omega t}$ and then take the real part of the solution.)

(c) Now you sing a song with $f(t)$ being a general function, show that

$$\psi(x,t) = \frac{1}{2\pi} \iint f(\tau) \csc(\frac{\omega L}{v}) \sin\frac{\omega x}{v} \cos \omega(t-\tau)\, d\omega d\tau.$$

(Hint: You may use Fourier transform to decompose $f(t) = \int \tilde{f}(\omega) e^{i\omega t} d\omega$ into superposition of sinusoidal waves and apply the result from (b).)

3. Transport an atom over macroscopic distance.

Paloma Ocola, now a Ph.D. student at Harvard, worked in our lab on a project aiming to transport a Cs atom over a macroscopic distance L =28 cm in $T = 0.25$ second. The idea is first confining the atom with mass m in an optical potential $V(x) = \frac{1}{2} m\omega^2 (x - x_0)^2$, where x is the atom position, x_0 is the center position of the potential, and then moving the atom by translating the potential by L.

For simplicity, we assume that the atom and the potential are initially at rest $x(0^-) = x_0(0^-) = x'(0^-) = x_0'(0^-) = 0$. The potential starts moving at $t = 0$ and stops at $t = T$, such that the potential is at rest again afterwards $x_0(t \geq T) = L$ and $x_0'(t \geq T) = 0$.

(a) In the frame co-moving with the potential the atom deviates from the trap center by $u \equiv x - x_0$. Show that the atom experiences a fictitious force and its motion in the moving frame is described by

$$u'' + \omega^2 u = -x_0''(t)$$

(b) A naïve transport scheme is to move the potential linearly with time $x_0(0 < t < L) = (L/T)t$. Show that the subsequent motion of the atom is

$$x(t \geq T) = L + \frac{2L}{\omega T} \sin\frac{\omega T}{2} \cos \omega(t - \frac{T}{2})$$

(Hint: You may use the Green's function method in HW4 question 2.)

(c) A better scheme developed by Paloma is to accelerate the potential for half the time $T/2$ and decelerate for time $T/2$. Show that the acceleration should be $a = \pm 4L/T^2$ and the subsequent motion of the atom is

$$x(t \geq T) = L + \frac{16L}{\omega^2 T^2} \sin^2\frac{\omega T}{4} \cos \omega(t - \frac{T}{2}).$$

(Comment: Paloma's scheme is superior since it induces much less excitations for experiments with $\omega T \gg 1$.)