Quantum state control of a single photon

A photon propagating in the z direction can be described by wavefunction
\[ |\psi> = \left( \psi_x \psi_y \right) e^{-i(kz-\omega t)}, \]
where \( \psi_x \) and \( \psi_y \) are the complex probability amplitudes to find the photon with polarization in the x and y directions, and \( |\psi_x|^2 + |\psi_y|^2 = 1 \). Some of the photon states are defined as
\[
\begin{align*}
\text{Horizontally polarized: } |H> &= (1,0) \\
\text{Vertically polarized: } |V> &= (0,1) \\
\text{Diagonally polarized: } |D> &= \frac{1}{\sqrt{2}} (1,1) \\
\text{Anti-diagonally polarized: } |A> &= \frac{1}{\sqrt{2}} (1,-1) \\
\text{Right circularly polarized: } |R> &= \frac{1}{\sqrt{2}} (1,-i) \\
\text{Left circularly polarized: } |L> &= \frac{1}{\sqrt{2}} (1,i)
\end{align*}
\]

These states are eigenstates of the Pauli matrices \( \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), \( \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \) and \( \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) and \( \vec{\sigma} \equiv (\sigma_x, \sigma_y, \sigma_z) \).

Conversions from an initial photon state \( |i> \) to a final state \( |f> = U|i> \) is described by unitary operators \( U \) that form a SU(2) group. One important property of the SU(2) group is that any operator \( U \) in the group can be expressed in terms of the scalar product of the Pauli vector \( \vec{\sigma} \) and a vector \( \vec{a} = (a_x, a_y, a_z) \) as \( U = e^{i(\phi + \vec{a} \cdot \vec{\sigma})} \). This is the homomorphic mapping between SU(2) and SO(3).

A. Determine \( U_1, U_2 \) and the associated vectors \( \vec{a}_1, \vec{a}_2 \) that perform the following transformations:
\[ |D> = U_1 |H> \text{ and } |L> = U_2 |A> \]

B. A quarter waveplate QWP is a birefringent material with orthogonal Fast and Slow axes. It transmits light and imposes different phase shifts \( \phi_F \) and \( \phi_S \) along the two axes such that \( \phi_S = \phi_F + \left( N + \frac{1}{4} \right) 2\pi \), where \( N = 0, \pm 1, \pm 2 \ldots \) is the order of the QWP. Consider a QWP with the Fast axis rotated w.r.t. the x direction by \( +\theta \), show that the unitary operator associated with a QWP is given by
\[
U_{QWP}(\theta) = e^{i\phi_F} \begin{pmatrix}
\cos^2 \theta + i \sin^2 \theta & (1 - i) \sin \theta \cos \theta \\
(1 - i) \sin \theta \cos \theta & \sin^2 \theta + i \cos^2 \theta
\end{pmatrix},
\]
And what is the associated $\vec{a}_{QWP}$?

Hint: Argue that $U_{QWP}(\theta) = R(\theta) \left( e^{i\phi_F} \begin{pmatrix} 0 & e^{i\phi_S} \\ e^{-i\phi_S} & 0 \end{pmatrix} \right) R(-\theta)$, where $R(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$ is the rotation matrix.

C. How does $U_{QWP}(\theta)$ rotate a Bloch vector $\vec{b} = \langle \vec{\sigma} \rangle$ on the Bloch sphere?

D. In quantum logic, standard single-qubit gates are

<table>
<thead>
<tr>
<th>Operator</th>
<th>Gate(s)</th>
<th>Matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pauli-X (X)</td>
<td>$\begin{pmatrix} 0 &amp; 1 \ 1 &amp; 0 \end{pmatrix}$</td>
<td></td>
</tr>
<tr>
<td>Pauli-Y (Y)</td>
<td>$\begin{pmatrix} 0 &amp; -i \ i &amp; 0 \end{pmatrix}$</td>
<td></td>
</tr>
<tr>
<td>Pauli-Z (Z)</td>
<td>$\begin{pmatrix} 1 &amp; 0 \ 0 &amp; -1 \end{pmatrix}$</td>
<td></td>
</tr>
<tr>
<td>Hadamard (H)</td>
<td>$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 &amp; 1 \ 1 &amp; -1 \end{pmatrix}$</td>
<td></td>
</tr>
<tr>
<td>Phase (S, P)</td>
<td>$\begin{pmatrix} 1 &amp; 0 \ 0 &amp; i \end{pmatrix}$</td>
<td></td>
</tr>
<tr>
<td>$\pi/8$ (T)</td>
<td>$\begin{pmatrix} 1 &amp; e^{i\pi/4} \ 0 &amp; e^{-i\pi/4} \end{pmatrix}$</td>
<td></td>
</tr>
</tbody>
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Show that at least 2 of them can be realized by few QWPs set at different angles.

Hint: In fact, any photon state can be prepared with just a few QWPs. Prove it if you are interested.