1. Coherent States

A simple harmonic oscillator (SHO) described by 
\[ H = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2}, \]
where \( p = -i\hbar \partial_q \), is a good model to understand coherent states. The general solution of a SHO is

\[ \psi(x, t) = \sum_{n=0} c_n e^{-iE_n t/\hbar} \psi_n(x), \]
where \( x = q\sqrt{\frac{m\omega}{\hbar}} \) is the dimensionless position, \( \psi_n(x) = \frac{e^{-x^2/2}}{\sqrt{2^n n!\sqrt{\pi}}} H_n(x) \) is the normalized \( n \)-th eigenstate with eigenenergy \( E_n = (n + \frac{1}{2}) \hbar \omega \) and \( \int \psi^*_n \psi_m = \delta_{nm} \), and the Hermit polynomials \( H_n(x) \) can be defined by the expansion of the generating function \( e^{-s^2+2sx} \equiv \sum_{n=0} \frac{1}{n!} H_n(x) s^n \).

A. A coherent state defined by \( a|A> = A|A> \) can be expressed in the basis of the eigenstates with coefficients \( c_n = \frac{A^n}{\sqrt{2^n n!}} \). Show that the wavefunction of the state \( |A> \) has the following close form

\[ \psi_A(x, t) = \langle x|A> = \pi^{-1/4} e^{-(x-A\cos \omega t)^2/2} e^{-i\phi(t)}, \]
\[ \phi(t) = \frac{\omega t}{2} + A(x - \frac{A}{2}\cos \omega t) \sin \omega t. \]

B. Show that in the limit \( A \to 0 \) the wavefunction reduces to the ground state \( \psi_0 \). For \( A \gg 1 \), show that the solution is a localized wave packet that closely follows the classical trajectory of a harmonic oscillator with minimal uncertainty \( \Delta x \Delta p = \hbar/2 \).

C. The coherent state has an interesting phase \( \phi \). Discuss the behavior of the phase for \( A = 0 \) (ground state) and \( A \gg 1 \) (classical limit). In particular, what is the physical interpretation of the phase for \( A \gg 1 \).

(Hint: Consider \( \bar{x} = A\cos \omega t \) is the classical position of the oscillator, \( \bar{p} = -A\hbar \sin \omega t \) is the classical momentum. You can see that the phase can be written as \( \phi = \frac{\omega t}{2} - \frac{\bar{x} \bar{p}}{\hbar} + \frac{\bar{x} \bar{p}}{2\hbar} \).)
2. Properties of coherent states
A photon field described by the Hamiltonian $H = \hbar \omega (a^+ a + \frac{1}{2})$ supports coherent states, which follows $a|\alpha >= \alpha |\alpha >$, where $a$ is the annihilation operator, $\alpha \in \mathbb{C}$ is the amplitude of the coherent state. Here we will study some of the most important properties of the coherent states.

A. Show that the definition of the coherent state is consistent with that based on the displacement operator, namely, $|\alpha >= D_\alpha |g >$, where $D_\alpha = e^{\alpha a^+ - \alpha^* a}$ translates the ground state $|\alpha = 0 >$ by $\alpha$. Your result should also be consistent with the definition in Question 1.

B. Show that any two coherent states have finite overlap:

$< \alpha |\beta > | = e^{-|\alpha - \beta|^2/2}$.

C. Show that coherent states $|\alpha >$ forms an over-complete basis, and can expand an arbitrary state $|\psi >$ as

$|\psi > = \frac{1}{\pi} \int |\alpha > < \alpha |\psi > d^2 \alpha$.

D. Show that the evolution of the coherent state gives

$U(t)|\alpha > = |\alpha e^{-i\omega t} >$,

where $U(t) = e^{-iHt/\hbar}$ is the operator that evolves the system forward in time by $t$.

3. Quantization of coherent states?
A bizarre thing about the coherent states $|\alpha >$ is that even they are eigenstates of the annihilation operator $a$, the associated eigenvalue $a \in \mathbb{C}$ can take an arbitrary value on the complex plane. In contrast, Fock states $|n >$ are eigenstates of the Hamiltonian $H = \hbar \omega (a^+ a + \frac{1}{2})$, but the eigenvalue $n = 0, 1, ...$ are discrete. Why are they so different?

A. Consider a truncated Hilbert space $|\psi > = \sum_{n=0}^3 c_n |n >$, write the annihilation operator $a$ in the Fock state basis as a $4 \times 4$ matrix. Show that there are only 4 “coherent states” in the truncated Hilbert space as the eigenstates of the matrix. What are the eigenvalues $\alpha$?

B. Compare the coherent states in the truncated Hilbert space vs those in the infinite Hilbert space $|\psi > = \sum_{n=0}^\infty c_n |n >$. How do coherent states form a continuum while Fock states do not?

(Hint: you may compute the eigenvalues and eigenvectors by a software. If the connection to the continuous coherent states is still unclear, try a bigger Hilbert space.)