1. Bose-Einstein distribution and Bose-Einstein condensation (BEC)
Consider \( N = \sum_{i=0} N_i \gg 1 \) ideal bosons with mass \( m \) are confined within a volume of \( V \) and \( n_i \) is the population in the \( i \)-th eigenstate. Given a grand canonical ensemble with chemical potential \( \mu \) and temperature \( T \), the probability to find \( \alpha = 0, 1, 2 \ldots \) bosons in the state is \( p_\alpha \propto e^{-\beta(\alpha E_i - \alpha n)} \) where \( \beta = \frac{1}{k_B T} \).

A. Show that the mean population \( \bar{N}_i = < \alpha p_\alpha > \) in the state is exactly given by the Bose-Einstein distribution \( \bar{N}_i = \frac{1}{e^{\beta E_i - \mu} - 1} \).

B. BEC occurs when a sizable fraction of the particles occupies the ground state \( N_0 = O(N) \). In this case the excited state population is given by \( N_{ex} = \sum_{i=1} N_i \). Show that the BEC transition occurs when \( \mu \) increases toward \( E_0 \).

(Comment: Here \( N_0 = O(N) \) means \( \lim_{N \to \infty} \frac{N_0}{N} > 0 \).)

C. As a concrete example, consider a large 3D box with volume \( V \) and \( E_0 \equiv 0 \). We may approximate the sum by an integral \( \sum_{i=1} N_i = \int_0^\infty n(E)\rho(E)\,dE \) and the 3D density of state is \( n(E) = \frac{V}{2\pi^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} \sqrt{E} \). Show that the equation of state is
\[
 n\lambda_{dB}^3 = n_0\lambda_{dB}^3 + g_3(e^\mu),
\]
Where \( n = \frac{N}{V} \) is the total particle density in the box, \( n_0 = \frac{N_0}{V} = e^\mu \left( 1 - e^\mu \right) \) is the condensate density, \( \lambda_{dB} = \hbar(2\pi m k_B T)^{-1/2} \) is the thermal de Broglie wavelength and \( g_n(z) = \frac{1}{\Gamma(n)} \int_0^\infty \frac{x^{n-1}}{z^{-1} e^{n-1}} \,dx \) is the Bose function, and BEC occurs when the phase density \( n_\phi \equiv n\lambda_{dB}^3 \) reaches \( n_\phi = g_3(1) \approx 2.612 \ldots \)

(Comment: Equation of state is the relation between thermodynamical variables, \( \mu \) and \( T \) in this case.)

D. Now consider \( N \gg 1 \) ideal bosons in a 2D box, show that BEC occurs only at \( T=0 \).

E. Now consider \( N \) bosons in a 3D isotropic harmonic potential \( H = \frac{p^2}{2m} + \frac{m \omega^2 r^2}{2} \) with eigenenergy \( E = (n_x + n_y + n_z)\hbar \omega, n_x, n_y, n_z = 0, 1, 2 \ldots \). Determine the BEC critical temperature \( T_c \) and the population in the ground state and first 3 excited states at \( T = T_c \).
2. Mean-field description of a Bose-Einstein condensate

Gross-Pitaevskii equation describes the wavefunction of a Bose condensate:

\[
\left( \frac{p^2}{2m} + V(r) + \frac{4\pi a^2}{m} |\psi(r)|^2 \right) \psi(r) = \mu \psi(r),
\]

where \( V(r) \) is the external potential, \( m \) is the boson mass, \( \mu \) is the chemical potential, \( a \) is the scattering length, and \( p = -i\hbar \nabla \) is the momentum operator. The condensate wavefunction is normalized as \( \int \psi^*(r)\psi(r)d^3r = N \), where \( N \) is the particle number.

Depending on the strength of the external potential \( V(r) \) and the scattering length \( a \), the BEC can be prepared in different regimes:

A. Non-interacting BEC with zero scattering length \( a = 0 \)
In this case, all atoms occupy the ground state of the harmonic potential. How would you write down the wavefunction of the system \( \Psi(r_1, r_2 ... r_N) \)?

B. Regular BEC with positive and sizable scattering length \( a > 0 \)
In this regime interaction \( < \frac{4\pi a^2}{m} \psi^*(r)\psi(r) > \) near the trap center is generally much greater than the kinetic energy \( < \frac{p^2}{2m} > \) and the kinetic energy term can be neglected. Many BECs are in this regime. Show that the particle density is \( n(r) = \frac{\mu - V(r)}{g} \) for position \( r \) satisfies \( V(r) < \mu \) and \( n(r) \to 0 \) for \( V(r) > \mu \). Given the normalization condition, determine the density profile \( n(r) = \psi^*(r)\psi(r) \) and chemical potential \( \mu \) in terms of \( N, a \) and \( \omega \) in a 3D harmonic potential \( V(r) = \frac{1}{2}m\omega r^2 \).

C. Attractive (unstable) BEC with negative scattering length \( a < 0 \)
Here BEC can be unstable against collapse. Assume the wavefunction is given by the Gaussian ansatz

\[
\psi(r) = Ae^{-r^2/4R^2},
\]

where \( R \) characterizes the rms size of the BEC. Calculate the expectation values of the kinetic, potential and interactions energies. Show that there is a critical scattering length \( a_c < 0 \) below which the BEC would collapse.

D. Low dimensional condensates
One way to experimentally study BECs in 1D or 2D is to confine bosons in a highly isotropic trap \( V(r) = \frac{1}{2}m(\omega_x x^2 + \omega_y y^2 + \omega_z z^2) \). One expects the BEC is two-dimensional when \( \omega_z \gg \omega_x \approx \omega_y \) and one-dimensional when \( \omega_z \ll \omega_x \approx \omega_y \). Here we consider the extreme regime, called quasi-2D BEC, realized when \( \omega_z \gg \frac{\mu}{\hbar} \gg \omega_x \approx \omega_y \equiv \omega_r \). Determine the atomic wavefunction \( \psi(r, z) \) in this limit and express the chemical potential \( \mu \) in terms of \( N, a, \omega_r \) and \( \omega_z \).
3. **Bogoliubov transformation**

The purpose of this question is to get an intuitive picture of the Bogoliubov transformation. Consider the Hamiltonian of two interacting fields \( \psi_a \) and \( \psi_b \).

\[
\hat{H} = \alpha (\hat{a}^+ \hat{a} + \hat{b}^+ \hat{b}) + \beta (\hat{a}^+ \hat{b}^+ + \hat{b} \hat{a}) ,
\]

where \( \hat{a} \) and \( \hat{b} \) are the Bosonic annihilation operators of the fields and thus they satisfy the commutation relationship: \([\hat{a}, \hat{a}^+] = [\hat{b}, \hat{b}^+] = 1 \) and \([a, b^+] = [b, a^+] = [a, b] = [a^+, b^+] = 0 \).

A. Write the Hamiltonian \( H \) in the matrix form in the Fock state basis \(|n_a, n_b\rangle\) of the fields. Here \( n_a, n_b = 0,1,2 \ldots \) and the particle numbers in the field. For simplicity, you may consider only the subspace with \( n_a = n_b \). Show that \( \alpha \) and \( \beta \) appear in the diagonal and off-diagonal terms.

B. Bogoliubov transformation is a unitary transformation that diagonalizes the Hamiltonian. It is based on the ansatz

\[
\begin{align*}
a^+ &= uc^+ + vd \\ b^+ &= ud^+ + vc
\end{align*}
\]

where \( c \) and \( d \) are annihilation operators of the new bosonic fields \( \psi_c \) and \( \psi_d \), which are super-positions of \( \psi_a \) and \( \psi_b \). We demand \([c, c^+] = [d, d^+] = 1 \) and thus the new particles are bosons as well. We can assume \( u \) and \( v \) are real and show that \( u^2 - v^2 = 1 \).

Show that with the proper choice of \( u \) and \( v \), the Hamiltonian can be diagonalized as

\[
H = \varepsilon (c^+ c + d^+ d) + \varepsilon - \alpha ,
\]

where \( \varepsilon = \sqrt{\alpha^2 - \beta^2} \).

C. The energy functional of the BEC is written in the form

\[
H = \frac{gn}{2} N + \frac{1}{2} \sum_{\vec{k} \neq 0} \left( \frac{\hbar^2 k^2}{2m} + gn_0 (a_{\vec{k}}^+ a_{\vec{k}} + a_{-\vec{k}}^+ a_{-\vec{k}}) + gn_0 (a_{\vec{k}}^+ a_{-\vec{k}}^+ + a_{-\vec{k}} a_{\vec{k}}) \right)
\]

Use the result of B and construct the quasi-particle operators \( b_{\vec{k}}, b_{-\vec{k}} \) from \( a_{\vec{k}}, a_{-\vec{k}}, a_{\vec{k}}^+, a_{-\vec{k}}^+ \) that diagonalize the Hamiltonian.