Physics 143A: Honors Waves, Optics, and Thermo  
Spring Quarter 2024  
Problem Set #3  
Due: 11:59 pm, Wednesday, April 10. Please submit to Canvas.

1. **(Math) Fourier expansion and Fourier transform exercise (6 points each)**
   You may Fourier expand a periodic function or Fourier transform a general function as

   **Fourier series expansion:** $y(x) = f(x + L) \equiv a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{2\pi nx}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{2\pi nx}{L}$

   **Fourier transform:** $y(x) = \int y(k)e^{ikx}dk$

   Determine the Fourier series of the following functions
   (a) Fourier expand $y(x) = |\sin x|$ 
   Hint: first determine the period of the function and then the basis you need to expand it.
   (b) Fourier expand an infinite series of periodic impulses $y(x) = \sum_{n \in \mathbb{Z}} \delta(x - n)$, where $\mathbb{Z} = \{0, \pm1, \pm2 ... \}$ includes all integers and $\delta(x)$ is Dirac’s Delta function.
   (c) Fourier transform a single impulse $y(x) = \delta(x)$
   (d) Fourier transform a series of random impulses $y(x) = \sum_{n} \delta(x - x_n)$.
   (e) Show that given $f(x) = \int f(k)e^{ikx}dk$, we have the following normalization condition $\int f^*(x)f(x)dx = 2\pi \int f^*(k)f(k)dk$
   Hint: Use the formula $\int e^{ikx}dx = 2\pi \delta(k)$

2. **Fourier series and transforms of a square wave (7 points each)**
   Consider a periodic function $f(t) = f(t + T)$, where $T$ is the period. See below

   ![Square wave diagram](image_url)

   We will expand it in 3 ways and compare the results.
   (a) Expand $f(t)$ with the trigonometric functions with the same period

   $$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos \frac{2\pi nt}{T} + b_n \sin \frac{2\pi nt}{T}).$$

   Determine $a_0, a_n$ and $b_n$.
   (b) Expand $f(t)$ with the exponential series with the same period
\[ f(t) = \sum_{n=-\infty}^{\infty} c_n e^{-\frac{it}{T}}. \]

Determine \( c_n \).

(c) Fourier transform \( f(t) \) over the entire range
\[ \tilde{f}(t) = \int f(t)e^{i\omega t}d\omega. \]

Determine \( \tilde{f}(\omega) \).

(d) All these expansions are equivalent. Show that \( c_n \) and \( \tilde{f}(\omega) \) can be expressed in terms of \( a_0, a_n \) and \( b_n \).

3. General solution of a driven harmonic oscillator (9 points each)

(a) A driven oscillator is described by equation of motion:
\[ x''(t) + \gamma x'(t) + \omega_0^2 x(t) = f(t) \]

Show that the particular solution is
\[ x(t) = \frac{1}{\frac{2\pi}{\omega_0} + \frac{1}{\pi} \sum_{n=1,3,5...} \frac{1}{n} \text{Im} \left( \frac{e^{i\omega_n t}}{\omega_0^2 - \omega_n^2 + i\gamma \omega_n} \right), \text{where } \omega_n = \frac{2\pi n}{T}. \]

(b) As an example, we consider the external force \( f(t) \) as described in question 2. Determine the explicit form of the solution of the oscillator.

Hint: \( x(t) = \frac{1}{2\omega_0} + \frac{1}{\pi} \sum_{n=1,3,5...} \frac{1}{n} \text{Im} \left( \frac{e^{i\omega_n t}}{\omega_0^2 - \omega_n^2 + i\gamma \omega_n} \right), \text{where } \omega_n = \frac{2\pi n}{T}. \)

4. Damped wave equation and guitar string (8 points each)

A guitar string moves according to the following damped wave equation
\[ \rho \partial_t^2 \psi(x, t) + b \partial_x \psi(x, t) = E \partial_x^2 \psi(x, t) \]

with the fixed boundary conditions
\[ \psi(0, t) = \psi(L, t) = 0 \]

(a) Determine the dispersion \( \omega(k) \) of the eigenmodes, where \( \omega \) is the angular frequency of the wave and \( k \) is the wavenumber.

(Hint: Dispersion is the relation between angular frequency and wave number. You may use the ansatz \( \psi = Ae^{ikx}e^{i\omega t} \) and show that the eigenfrequency is \( \omega = Re[\omega] = \sqrt{\frac{E}{\rho} k^2 - \frac{b^2}{4\rho^2}}. \)

Next show that to satisfy the boundary condition the wavefunction should have the form \( \psi = A \sin kx e^{i\omega t} \) and the wavenumber \( k = k_n = \frac{(n+1)\pi}{L} \) can only take discrete values with \( n=0,1,2,3... \).

(b) Assume the initial position and velocity of the string are given by
\[ \psi(x,0) = f(x), \quad \partial_t \psi(x,0) = g(x) \]

Consider a frictionless string \( b = 0 \) for simplicity, determine the wavefunction \( \psi(x, t) \) for \( t > 0 \) and verify that the solution is always real.

(c) A string instrument can produce a rich spectrum with fundamental \( \omega_0 \) and unique overtones \( \omega_n, n = 1, 2 \ldots \). Without dissipation \( b = 0 \) the pitch of the overtone is \( \omega_n = (n + 1)\omega_0 \). Show that in the presence of small damping \( 0 < b \ll E\rho/\ell^2 \), the overtones are detuned from the fundamental as \( \omega_n = (n + 1)\omega_0 + \delta_n \). For small damping, show that

\[ \delta_n \approx (n + 1 - \frac{1}{n + 1}) \frac{b^2}{8\rho^2\omega_0} \]

Such effect is more severe for in the range of low frequency.

(Hint: in the presence of damping, the fundamental frequency is also shifted.)