1. **Anti-reflective (AR) coating**

Lord Rayleigh made a surprising discovery in the 1880s that an aged glass transmits more light than does a brand new glass! This, he found out, is due to a thin layer of tarnish on the surface of the old glass. Such coating layer results in suppression of reflections. Nowadays AR coating is everywhere in eye glasses, cameras, telescopes and other optical instruments. We will investigate how it works.

Consider a light beam reaches two interfaces from $n_i$ to $n_c$ to $n_s$. Here we assume $n_i < n_c < n_s$. Two reflections from the interfaces are shown and are the only ones we will consider. The wavelength of the light in vacuum is $\lambda_0$.

A. Show that the phase difference between the two reflections at the observing plane $P$ is given by

$$\Delta \phi = \frac{4\pi n_c L}{\lambda_0} \cos \theta_c$$

B. Assume $\lambda_0 = 500$ nm, the incident angle is $\theta_i = 45^\circ$, what is the minimal thickness $L$ to best reduce the reflection?

C. Given $n_i = 1.0003$ (air) and $n_s = 1.47$ (Pyrex), show that perfect destructive interference can be reached for small angles $\theta_i, \theta_c, \theta_s \ll 1$ if we choose the coating material with index of refraction $n_c \approx 1.212$.

Hint: you may use the Fresnel formulas to calculate the reflection and transmission amplitudes (for s-polarization):

- **Reflection fraction:**
  $$\frac{E_{\text{ref}}}{E_{\text{inc}}} = \frac{n_1 \cos \theta_i - n_2 \cos \theta_t}{n_1 \cos \theta_i + n_2 \cos \theta_t}$$

- **Transmission fraction:**
  $$\frac{E_{\text{tra}}}{E_{\text{inc}}} = \frac{2n_1 \cos \theta_i}{n_1 \cos \theta_i + n_2 \cos \theta_t}$$

Note that these equations are identical to the reflection and transmission ratios of string waves at the interface for small angles $\theta_i$ and $\theta_c$. 
2. Ideal gas (10 points each)

The laws we learned about the ideal gas are \( PV = NkT \) and internal energy of the gas \( U = \gamma NkT \), where \( \gamma = 3/2 \) for an atomic gas and \( \approx 5/2 \) for a molecular gas. We can derive many other thermal properties of the gas.

a) Heat capacity:

Given the isochoric specific heat \( c_V = \frac{1}{m} \partial_T Q|_V \), show that we can write the internal energy of the gas as

\[ U = c_V NmT, \]

where \( m \) is the atomic/molecular mass.

Hint: \( \Delta U = \Delta Q - P \Delta V \) and \( \partial_T Q|_V \) means \( \Delta Q / \Delta N \) when volume \( V \) is unchanged \( \Delta V = 0 \).

b) Compressibility:

In the derivation of sound speed few weeks ago, we introduced the compressibility as the fractional reduction of the volume per unit pressure added to the system:

\[ \beta = -\frac{1}{V} \frac{\partial V}{\partial P} \]

We are now in good position to derive it. Two forms of \( \beta \) can we determine: isothermal and isentropic compressibility

\[ \beta_T = -\frac{1}{V} \left( \frac{\partial V}{\partial P} \right)_T = \frac{1}{p} \quad \text{and} \quad \beta_S = -\frac{1}{V} \left( \frac{\partial V}{\partial P} \right)_S = \frac{\gamma}{(\gamma+1)p}. \]

Show that they are \( \beta_T = \frac{1}{p} \) and \( \beta_S = \frac{\gamma}{(\gamma+1)p} \) for an ideal gas.

Here isothermal means the temperature is kept constant, while isentropic means the process is adiabatic and reversible. Which \( \beta \) should we use to estimate the speed of sound \( v = 1/\sqrt{n\beta} \)? It turns out isentropic compressibility \( \beta_S \) gives a better estimation. Give your argument why it is more reasonable to use \( \beta_S \) than \( \beta_T \)?

(Hint: when sound wave shakes the molecules, will there be heat flowing between molecules? If yes, temperature could be a constant, if not, the process is adiabatic.)

c) Thermal expansion:

Thermal expansion determines the fractional change of the system when the temperature increases by 1 unit. There are again two possible processes: isobaric thermal expansion coefficient \( \alpha_V = \frac{1}{V} \left( \frac{\partial V}{\partial T} \right)_P \) and isochoric coefficient \( \alpha_p = \frac{1}{p} \left( \frac{\partial P}{\partial T} \right)_V \). Show that both lead to the same result as \( \alpha_V = \alpha_p = \frac{1}{T} \).

d) How much does the mean molecular distance in an ideal gas increases fractionally when temperature increases from 300K to 301K?

Hint: You may use \( \Delta V = \partial_T V \Delta T \) for small \( \frac{\Delta T}{T} = \frac{1}{300} \ll 1 \).
3. Maxwell-Boltzmann distribution (10 points each)

A great insight from Maxwell is that the velocity distribution of molecules in an ideal gas with mass \( m \) and velocity \( \vec{v} = (v_x, v_y, v_z) \) is given by

\[
P(\vec{v}) = p(v_x)p(v_y)p(v_z),
\]

where \( p(v_i) \propto e^{-mv_i^2/2kT} \) is the Maxwell-Boltsmann distribution.

a) Given the probability conservation \( \int_{-\infty}^{\infty} p(x)dx = 1 \) and \( \iiint P(\vec{v})d^3v = 1 \) for any stochastic variable \( x \), determine the explicit form of \( p(v_x) \) and \( P(\vec{v}) \).

b) Show that the probability distribution of \( P(v) \), where \( v = |\vec{v}| \geq 0 \) is the absolute value of the molecular velocity, is given by

\[
p(v) = \left(\frac{m}{2\pi kT}\right)^{3/2} \frac{4\pi v^2}{4} e^{-mv^2/2kT},
\]

which is the most common form of Maxwell-Boltzmann distribution.

Hint: A coordinate transform from Cartesian coordinate \((x, y, z)\) to spherical coordinate should satisfy probability conservation. Assume the probability to find a particle in space within a small volume element \( dv \) is \( dP = P(\vec{r})d\vec{v} \), the same probability should be found in a new coordinate system, namely,

\[
dP = P(\vec{r})dxdydz = P(\vec{r})r^2 \sin\theta drd\theta d\phi \equiv P(r, \theta, \phi)drd\theta d\phi.
\]

a) Determine the root-mean-square velocity \( v_{rms} = \sqrt{\langle v^2 \rangle} \) and show that \( v_{rms} = \sqrt{3kT/m} \) is consistent with the equipartition theorem. Show that the rms velocity is higher than the isothermal or isentropic sound velocity \( v_p = \frac{1}{\sqrt{n\beta}} \). Does it make sense that air molecules can be supersonic?
4. Refrigeration (10 points each)

Refrigerator is a device that receives energy (electricity) from the power plant in order to remove energy from the cold reservoir (food in the fridge) at a lower temperature $T_c$ and deliver the energy into the hot reservoir (atmosphere) at a higher temperature $T_H$.

![Diagram of energy flow in a refrigerator]

a) Compare the above process with an engine, we find that the above process is essentially a Carnot cycle running in reverse. Determine how much energy $W$ is demanded in order to extract one Joule of energy from the cold source.

Hint: Calculate $\frac{\Delta W}{\Delta Q_c}$ according to the definition in the diagram. You may just use the results we derived in Lectures 22-23 and consider the processes running backwards. Show the result in terms of the temperatures $T_H$ and $T_C$.

b) Your fridge keeps the food at around 40 F, while the ambient temperature is 72 F. How many Joules of electricity are needed to remove 1 Joule of energy from the food in the fridge?

c) How far can we cool? We can now cool atoms and molecules to nano-Kelvins. Show that if we treat the atoms as the heat source at $T_C = T_H = 300$ K and the lab as the heat sink, the amount of energy required to cool an atom from $T_C = T_H = 300$ K to $T_C = 10^{-9}$ K using the reverse Carnot cycle would be

$$W = -c_v k \left(T_H \left(1 - \ln \frac{T_H}{T_C}\right) - T_C\right)$$

d) Show that you need infinite amount of energy to cool a system from any finite temperature to zero temperature. The divergence goes logarithmically in the limit of $T_C = 0$ as $W = c_v k T_H \ln \frac{T_H}{T_C}$.

Hint: This result demonstrates that zero temperature is not attainable.