a. The light ray inside the medium with refractive index $n_c$ travels an extra distance

$$l_1 = \frac{2L}{\cos \theta_c}$$

And the one outside travels an extra distance

$$l_2 = 2L \tan \theta_c \sin \theta_i$$

Path difference is related to the phase difference by the wavelength in that medium

$$\delta = \frac{2\pi l}{\lambda_n} = \frac{2\pi}{\lambda_0}nl$$

where $\lambda_n = \lambda/n$ is the wavelength in a medium with refractive index $n$. So the total phase difference is

$$\Delta \phi = \delta_1 - \delta_2$$

$$= \frac{2\pi}{\lambda_0} (n_c l_1 - n_i l_2)$$

$$= \frac{4\pi L}{\lambda_0} \left( \frac{n_c}{\cos \theta_c} - n_i \tan \theta_c \sin \theta_i \right)$$

$$= \frac{4\pi L}{\lambda_0} \left( \frac{n_c}{\cos \theta_c} - n_c \tan \theta_c \sin \theta_c \right)$$

$$= \frac{4\pi n_c L}{\lambda_0} \cos \theta_c$$

where in the third line we used Snell’s law $n_i \sin \theta_i = n_c \sin \theta_c$.

b. We are given that $\lambda_0 = 500$ nm and $\theta_i = 45^\circ$. To reduce reflection, we would want $\Delta \phi = (2m + 1)\pi$, ($m = 0, 1, 2, \cdots$). So

$$L_{\text{min}} = \frac{\lambda_0}{4n_c \cos \theta_c}$$

When $\theta_i = 45^\circ$, from Snell’s law

$$\sin \theta_c = \frac{n_i}{n_c} \sin \theta_i = \frac{1}{\sqrt{2} n_c}$$
And so
\[ L_{\text{min}} = \frac{\lambda_0}{4n_c} \left( 1 - \frac{1}{2} \frac{n_i^2}{n_c^2} \right)^{1/2} \]  
(7)

**c.** Given that the angles are small, we can use the small angle approximation. So the Fresnel equations become

\[
\begin{align*}
\frac{E_{\text{ref}}}{E_{\text{in}}} &= \frac{n_1 - n_2}{n_1 + n_2} \\
\frac{E_{\text{tra}}}{E_{\text{in}}} &= \frac{2n_1}{n_1 + n_2}
\end{align*}
\]
(8)

Now for perfect destructive interference, we would want the amplitudes of all the light rays to be equal. In our case, we would have two light rays-

- Light ray reflected from the \( n_i - n_c \) interface: \( E_1 \)
- Light ray transmitted through the \( n_i - n_c \) interface, reflected off the \( n_c - n_s \) interface and then transmitted again from the \( n_c - n_i \) interface: \( E_2 \).

Using the Fresnel formulas, we see that

\[ E_1 = \frac{n_i - n_c}{n_i + n_c} E_{\text{in}} \]  
(9)

And

\[ E_2 = \left( \frac{2n_c}{n_i + n_c} \right) \left( \frac{n_c - n_s}{n_c + n_s} \right) \left( \frac{2n_i}{n_i + n_c} \right) E_{\text{in}} \]  
(10)

And so the condition for equal amplitudes

\[ \frac{n_i - n_c}{n_i + n_c} = \frac{4n_in_c}{(n_i + n_c)^2} \left( \frac{n_c - n_s}{n_c + n_s} \right) \]  
(11)

Solving for \( n_c \) gives \( n_c \approx 1.212 \).

2.

**a.** Given that isochoric specific heat is calculated at fixed volume, we see from the relation between internal energy and heat that

\[ c_V = \frac{1}{M} \frac{\partial T}{\partial V} |_{V} = \frac{1}{M} \frac{\partial T}{\partial V} |_{V} = \frac{1}{m} \gamma Nk \]

\[ \implies \gamma Nk = Mc_v \]  
(12)

where we used \( U = \gamma NkT \). So

\[ U = \gamma NkT = Mc_vT = mNc_vT \quad (M = mN) \]  
(13)
b. For isothermal compressibility, we see that
\[
\beta_T = -\frac{1}{V} \left( \frac{\partial V}{\partial P} \right)_T = -\frac{1}{V} \frac{\partial}{\partial P} \left( \frac{NkT}{P} \right) = \frac{1}{V} \left( \frac{NkT}{P} \right)^2 = \frac{1}{P} \tag{14}
\]

To calculate isoentropic compressibility, we first note that
\[
dU = TdS - PdV \\
\implies PdV = -dU \quad \text{(for isoentropic case i.e. } dS = 0) \\
PdV = -\gamma PdV - \gamma VdP \quad \text{(using } U = \gamma NkT = \gamma PV) \\
\frac{\partial V}{\partial P} = -\frac{\gamma}{(1 + \gamma)} \frac{V}{P} \tag{15}
\]
And so \( \beta_S = \frac{\gamma}{(1 + \gamma)} \frac{1}{P} \).

c. We can use the ideal gas equation to calculate the thermal expansion coefficients.
\[
\alpha_V = \frac{1}{V} \left( \frac{\partial V}{\partial T} \right)_P = \frac{1}{V} \left( \frac{\partial}{\partial T} \frac{NkT}{P} \right)_P = \frac{Nk}{PV} = \frac{1}{T} \\
\alpha_P = \frac{1}{P} \left( \frac{\partial P}{\partial V} \right)_T = \frac{1}{P} \left( \frac{\partial}{\partial V} \frac{NkT}{V} \right)_V = \frac{1}{T} \tag{16}
\]
d. The mean distance is given by \( l = \left( \frac{V}{N} \right)^{1/3} = \left( \frac{kT}{m} \right)^{1/3} \). And so,
\[
dl = \frac{1}{3} \left( \frac{k}{P} \right)^{1/3} T^{1/3} dT \\
dl = \frac{1}{3} \frac{dT}{T} = \frac{1}{900} \tag{17}
\]

3.

a. Let \( p(v_i) = N \exp(-mv_i^2/2kT) \). Using probability conservation, we get
\[
\int_{-\infty}^{\infty} dv_i p(v_i) = 1 \\
N \int_{-\infty}^{\infty} dv_i \exp \left( -\frac{mv_i^2}{2kT} \right) = 1 \\
N \sqrt{\frac{2\pi kT}{m}} = 1 \\
N = \sqrt{\frac{m}{2\pi kT}} \tag{18}
\]
Similarly, for \( P(V) = p(v_x)p(v_y)p(v_z) = \left( \frac{m}{2\pi kT} \right)^{3/2} \exp \left( -m(v_x^2 + v_y^2 + v_z^2)/2kT \right) \)

b. We can transform to spherical coordinates
\[
P(\vec{v})d\nu d\varphi = P(\vec{v}) |\vec{v}|^2 \sin \theta dr d\theta d\phi \tag{19}
\]
where we can also write $P(\vec{v}) = P(|v|, \theta, \phi)$ in spherical coordinates. Integrating over the angular parts will give us $P(|v|)$. So

$$P(|v|) = \int d\theta \int d\phi P(\vec{v})|v|^2 \sin \theta = \left(\frac{m}{2\pi kT}\right)^{3/2} 4\pi v^2 e^{-mv^2/2kT}$$  \hspace{1cm} (20)

c. Let us first calculate $\langle v^2 \rangle$

$$\langle v^2 \rangle = \int v^2 P(v) dv = 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} \int_0^\infty dv v^4 e^{-mv^2/2kT}$$

$$= \frac{3kT}{m}$$

And so $v_{rms} = \sqrt{\frac{3kT}{m}}$. This is consistent with the equipartition theorem. From equipartition theorem, we have $\frac{1}{2}kT$ amount of energy associated with each degree of freedom

$$E = 3 \times \frac{1}{2}kT = \frac{1}{2}mv^2$$

$$\implies v = \sqrt{\frac{3kT}{m}}$$  \hspace{1cm} (22)

The isoentropic velocity is

$$v_p = \frac{1}{\sqrt{\rho \beta'_s}} = \frac{1}{\sqrt{\frac{\gamma}{m}} \frac{\gamma + 1}{\gamma}} \sqrt{P} = \frac{\gamma + 1}{\gamma} \frac{\sqrt{kT}}{m} = \sqrt{\frac{5/3kT}{m}} < \sqrt{\frac{3kT}{m}}$$  \hspace{1cm} (23)

4.

a. The energies extracted and received are proportional to the temperatures. The net work done by the refrigerator is $E_H - E_C$ while the energy put in is $E_C$. So the efficiency is

$$\frac{E_H - E_C}{E_C} = \frac{T_H - T_C}{T_C} = \frac{T_H}{T_C} - 1$$  \hspace{1cm} (24)

b. We found above

$$\frac{\Delta W}{\Delta Q_C} = \frac{T_H}{T_C} - 1$$

$$\implies \Delta W = \Delta Q \left(\frac{T_H}{T_C} - 1\right)$$  \hspace{1cm} (25)

So converting the values of temperatures to Kelvin and using $\Delta Q = 1$ J, we see that

$$\Delta W = \frac{295.6}{277.6} - 1 = 0.065 J$$  \hspace{1cm} (26)
c. Using \( \Delta Q = mc_v \Delta T \), we can write

\[
dW = dQ \left( \frac{T_H}{T} - 1 \right)
= mc_v \left( \frac{T_H}{T} - 1 \right) dT
\]

\[
\Rightarrow W = mc_v \int_{T_H}^{T_C} \left( \frac{T_H}{T} - 1 \right) dT
= -mc_v \left( T_H \left( 1 - \ln \frac{T_H}{T_C} \right) - T_C \right)
\]

(27)

d. We can see from above that as \( T_C \to 0 \), the expression for work diverges logarithmically.

\[
\lim_{T_C \to 0} W = mc_v T_H \ln \frac{T_H}{T_C} \to \infty
\]

(28)