Few remarks on wave eqn and Fourier transform

\[ \frac{\partial^2 \psi}{\partial x^2} = v^2 \frac{\partial^2 \psi}{\partial t^2} \]

General solution is \( \psi = f(x-vt) + g(x+vt) \)

We may decompose \( f(x) \) and \( g(x) \) in Fourier space

\[ f(x) = \int f(k) e^{ikx} dk \Rightarrow f(x-vt) = \int f(k) e^{ik(x-vt)} dk \]

Initial condition \( \psi(x,0) = \int f(k)e^{ikx} + g(k)e^{ikx} dk \)

Late time solution: \( \psi(x,t) = \int f(k) e^{ik(x-vt)} + g(k) e^{ik(x+vt)} dk \)

What if the wave is propagating in other directions?

Wave number \( \mathbf{k} = (k_x, k_y) \) is a vector in the propagation direction.

\[ \Rightarrow e^{i(\mathbf{k} \cdot \mathbf{r} - vt)} \text{ describes wave moving in } \mathbf{k} \text{ with wave # } |\mathbf{k}| = \frac{2\pi}{\lambda} \]

\[ \Rightarrow \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \]

in 2-dimensions.

Double check: \( \frac{\partial^2 \psi}{\partial x^2} = \omega^2 \psi \)

\[ \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = (\cos \theta + \sin \theta) k^2 \psi \]

\[ \Rightarrow \omega^2 = v^2 k^2 \]
Wave phenomena: longitudinal and transverse waves.

**Longitudinal**

- Propagation: Displacement along the propagation.
- Examples: sound, spring, seismic P-wave.

**Transverse**

- Displacement transverse to the propagation.
- Examples: Strings, EM waves, seismic S-wave, water waves.

Strings

\[ T(x, y) \text{ is along the string } T_y/T_x = \psi'(x) \]

\[ T(x-\Delta x, y) = -(T_x, T_y) \psi'(x-\Delta x/2) \]
\[ T(x+\Delta x, y) = (T_x, T_y) \psi'(x+\Delta x/2) \]

No transverse motion: \( \sum T_x = T_x - T_y = 0 \)

Yes longitudinal motion: \( \sum T_y = T \psi'(x+\Delta x/2) - T \psi'(x-\Delta x/2) = \rho \Delta x \dot{\psi'} \)

\[ \Rightarrow \rho \Delta x \dot{\psi'} = T (\psi'(x+\Delta x/2) - \psi'(x-\Delta x/2)) \]
\[ = T \Delta x \dot{\psi'} \]

\[ \Rightarrow \rho \dot{\psi'} = T \dot{\psi'} \text{ Compared with } \dot{\psi'} = v^2 \ddot{\psi'} \]

\[ \Rightarrow \text{String velocity } v = \frac{\omega}{k} = \sqrt{\frac{T}{\rho}} \]

If length is fixed \( k = \text{const.} \)

- Higher tension: higher pitch \( \omega \propto \sqrt{T} \)
- Higher density: lower pitch \( \omega \propto \sqrt{\rho} \)
Boundary conditions

1. Fixed \( \varphi(x=0) = \varphi(x=L) = 0 \)

\[
\varphi(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L}
\]

How about time dependence? Each wave goes like \( e^{\pm ikx \pm iwt} \) with \( v = c/k \) given by \( \sqrt{T/p} \)

\[
\varphi(x,t) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L} (b_n e^{ikvt} + c_n e^{-ikvt})
\]

2. Open boundary condition

Another way to write the solution

\[
\varphi = \sum_{n=0}^{\infty} a_n \sin k_n x \cos (\omega_n t + \phi_n)
\]

\[
L = \frac{1}{4} \lambda, \frac{3}{4} \lambda, \frac{5}{4} \lambda \ldots
\]

\[
k_n = \frac{\pi}{\lambda} = \frac{\pi}{2L} \frac{2n+1}{4}
\]

3. Free boundary condition

\[
\varphi = \sum_{n} a_n e^{ik_n(x-ut)} + b_n e^{ik_n(x+ut)} \rightarrow \int \varphi(k)e^{ik(x-ut)} + \varphi(k)e^{ik(x+ut)} dk
\]

forward waves backward waves

4. Connecting two strings of diff. density \( \rho_1, \rho_2 \)

\[
\rho_1 \quad \rho_2
\]

Waves going to diff. medium

water. light. seismic and others.

What do we do at the boundary?